Convergence of the supercell method for computation of defect modes in one-dimensional photonic crystals

Seungil Kim[a,*], Tae In Kwon[b,1]

[a]Department of Mathematics and Research Institute for Basic Sciences, Kyung Hee University, Seoul 130-701, Korea
[b]Department of Mathematics, Changwon University, Changwon 641-773, Korea

Abstract

In this paper, we analyze the spectrum induced by the supercell method for studying locally defected one-dimensional photonic crystals. By using the propagation matrix method, we show that the continuous spectrum of the periodic structure of the supercell converges to that of the defected photonic crystal. Also, it can be shown that frequencies of localized defected modes in the bandgap are included in a narrow interval (whose length diminishes exponentially with increasing size of the supercell) of the continuous spectrum of the supercell method.

Keywords: photonic crystal, supercell method, propagation matrix, spectrum, Floquet-Bloch theory
2000 MSC: 35Q60, 35P25

1. Introduction

One dimensional photonic crystals are periodically layered structures composed of different dielectric materials and it is well-known that they have a special spectral structure, so-called photonic bandgap, i.e., wave fields of frequencies in the bandgap are prohibited from propagating in the periodic structure due to multiple scattering and destructive interference [1, 4]. When a local defect is introduced to perfect photonic crystals, localized resonance modes may appear at frequencies inside the bandgap. This attractive phenomenon can be realized in artificial structures to confine the optical energy in the defected region. One of effective numerical techniques for calculating bandgap structures and localized resonance modes of defected photonic crystals are the supercell method, which takes a finite-sized periodic structure including defects (called a supercell), rearranges the supercell periodically in the whole region and hence recovers the periodicity. Therefore the resulting periodicity of the supercell method allows Floquet-Bloch theory and plane wave expansion analysis for computing the bandgap structure. The basic idea of the supercell method is based on the inspiration that if the size of the supercell is large enough, then the interaction between localizations can be ignorable since resonance modes are highly localized near the defect and decay rapidly [7]. The convergence of the supercell method in $\mathbb{R}^2$ was investigated in [7] based on the Green's function technique. In this paper, we will use the propagation matrix method to establish the same convergence result of the supercell method in $\mathbb{R}$ and describe the spectrum induced by the supercell method and approximation of defect modes. It is also worth noting that there are other computational techniques for finding localized modes such as PML [5] and propagation matrix method [6], which remodel defected infinite periodic structures into finite-sized periodic structures with defects supplemented with the radiation condition at infinity.

*Corresponding author
Email addresses: sikim@khu.ac.kr (Seungil Kim), taekwon@changwon.ac.kr (Tae In Kwon)

[1]This research is financially supported by Changwon National University in 2015∼2016.

2. Spectrum of photonic crystals with a defect

We first consider the perfectly periodic problem

$$(\frac{1}{\varepsilon}u')' + k^2 u = 0 \text{ in } \mathbb{R}, \tag{2.1}$$

where $'$ stands for differentiation with respect to x and ε is the electric permittivity, a piecewise constant function defined by

$$\varepsilon(x) = \begin{cases} \varepsilon_1 \text{ for } 0 < x < L_1, \\ \varepsilon_2 \text{ for } L_1 < x < L_1 + L_2 \end{cases}$$

with $\varepsilon(x+L) = \varepsilon(x)$. Here, $L = L_1 + L_2$ is the period of the unit cell of the photonic crystal. Solutions to the problem (2.1) can be investigated by the propagation matrix method as follows. Noting that both wave fields u and fluxes $\frac{1}{\varepsilon}u'$ are continuous on interfaces between two different dielectric layers, we introduce two component wave field vectors $U = (u, \frac{1}{\varepsilon}u')^t$. Then it can be easily shown that for given U at $x = 0$ solutions at $x = L_1$ are evaluated by $P_{L_1}U$ for $k > 0$, where $P_{L_1} = P_{L_1}(k)$ is the 2×2 propagation matrix

$$P_{L_1} = \begin{bmatrix} \cos(\mu_1) & \frac{\sqrt{\varepsilon_1}}{k}\sin(\mu_1) \\ -\frac{k}{\sqrt{\varepsilon_1}}\sin(\mu_1) & \cos(\mu_1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\sqrt{\varepsilon_1}}{k} \end{bmatrix}^{-1} \begin{bmatrix} \cos(\mu_1) & \sin(\mu_1) \\ -\sin(\mu_1) & \cos(\mu_1) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\sqrt{\varepsilon_1}}{k} \end{bmatrix} \tag{2.2}$$

with $\mu_1 = k\sqrt{\varepsilon_1}L_1$. Although ε_1 is also a variable of P_{L_1}, the dependence of P_{L_1} on ε_1 will be omitted in the notation for simplicity. The propagation matrix $P_{L_2} = P_{L_2}(k)$ for the second dielectric material with the permittivity ε_2 is analogously defined and the propagation matrix for the unit cell of length L can be written as

$$\begin{aligned} P_L = P_{L_2}P_{L_1} &= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \\ &= \begin{bmatrix} \cos(\mu_1)\cos(\mu_2) - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\sin(\mu_1)\sin(\mu_2) & \frac{1}{k}\big(\sqrt{\varepsilon_1}\sin(\mu_1)\cos(\mu_2) + \sqrt{\varepsilon_2}\cos(\mu_1)\sin(\mu_2)\big) \\ -k\big(\frac{1}{\sqrt{\varepsilon_1}}\sin(\mu_1)\cos(\mu_2) + \frac{1}{\sqrt{\varepsilon_2}}\cos(\mu_1)\sin(\mu_2)\big) & \cos(\mu_1)\cos(\mu_2) - \frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}}\sin(\mu_1)\sin(\mu_2) \end{bmatrix} \end{aligned}$$

with $\mu_2 = k\sqrt{\varepsilon_2}L_2$. When $k = 0$, a straightforward computation shows

$$P_L = \begin{bmatrix} 1 & \varepsilon_1 L_1 + \varepsilon_2 L_2 \\ 0 & 1 \end{bmatrix}.$$

By invoking the formula (2.2) of P_{L_j} (similar to a rotation) we can see that $P^{\theta}_{L_j}(k) = P_{\theta L_j}(k)$ for $-1 \le \theta \le 1$ (see e.g. [3, Chap. 6]), and the following lemma is easily verified by a simple computation.

Lemma 2.1. *Let V be an eigenvector of P_L for an eigenvalue λ. For $0 \le \theta \le 1$, $P^{1-\theta}_{L_1}P_{L_2}P^{\theta}_{L_1}$ has the eigenvalue λ for an eigenvector $P^{1-\theta}_{L_1}V$. Analogously, $P^{1-\theta}_{L_2}P_{L_1}P^{\theta}_{L_2}$ has the eigenvalue λ for an eigenvector $P^{-\theta}_{L_2}V$.*

Now, the trace of P_L is given by

$$\mathrm{Tr}(P_L) = 2\cos(\mu_1)\cos(\mu_2) - \left(\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}} + \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\right)\sin(\mu_1)\sin(\mu_2).$$

In addition, since $\det(P_L) = 1$, eigenvalues λ_1 and λ_2 of P_L are either complex numbers of magnitude 1 and $\lambda_1 = \bar{\lambda}_2$ when $|\mathrm{Tr}(P_L)| \le 2$ or real numbers such that $|\lambda_1| > 1 > |\lambda_2|$ when $|\mathrm{Tr}(P_L)| > 2$. The analysis on the spectrum of the problem (2.1) in terms of $\mathrm{Tr}(P_L)$ can be found in [1, 6] and we discuss it here for extending the idea to the supercell method. To this end, we first note that as the operator involved in the problem (2.1) is self-adjoint, the spectrum of the problem is real. Let $\sigma_c = \{k^2 \in \mathbb{R}_+ \,:\, |\mathrm{Tr}(P_L(k))| \le 2\}$, where $\mathbb{R}_+ = \{x \in \mathbb{R} \,:\, x \ge 0\}$. From here on we assume that frequencies k are non-negative.

For $k^2 \in \sigma_c$, there exists an eigenvalue λ_1 of P_L with $|\lambda_1| = 1$ associated with an eigenvector U_1. We can find the wave field vector $U(x)$ of the solution u to (2.1) such that $U(0) = U_1$. It can be written as $U(x) = P_{L_1}^{x/L_1} U_1$ for $0 \le x \le L_1$ and $U(x) = P_{L_2}^{(x-L_1)/L_2} P_{L_1} U_1$ for $L_1 \le x \le L$ in the unit cell. Since we have that by the definition of the propagation matrix

$$U(x+nL) = \begin{cases} (P_{L_1}^{1-\theta} P_{L_2} P_{L_1}^{\theta})^n U(x) \text{ with } \theta = (L_1 - x)/L_1 & \text{for } 0 \le x \le L_1, \\ (P_{L_2}^{1-\theta} P_{L_1} P_{L_2}^{\theta})^n U(x) \text{ with } \theta = (L - x)/L_2 & \text{for } L_1 \le x \le L, \end{cases}$$

Lemma 2.1 yields that $U(x+nL) = \lambda_1^n U(x)$ for $0 \le x \le L$ and $n \in \mathbb{Z}$, from which it can be concluded that there exists a non-trivial bounded solution for $k^2 \in \sigma_c$. Noting that $U(L) = P_L U(0) = \lambda_1 U(0)$ and $\lambda_1 = e^{i\tau}$ for $0 \le \tau \le 2\pi$, the set σ_c also can be interpreted as the union of eigenvalues of the problem in the unit cell for $\tau \in [0, 2\pi]$

$$(\frac{1}{\varepsilon} u')' + k^2 u = 0 \text{ in } (0, L),$$
$$U(0) = e^{i\tau} U(L).$$

In case of k^2 such that $|\mathrm{Tr}(P_L(k))| > 2$, there are two distinct real eigenvalues λ_1 and λ_2. Let U_1 and U_2 be eigenvectors of P_L for λ_1 and λ_2, respectively. Since the wave field vector of solutions to the problem (2.1) is of the form $U = \alpha_1 U_1 + \alpha_2 U_2$ at $x = 0$ for some constants α_1 and α_2, we have $U(x) = \alpha_1 \lambda_1^n U_1 + \alpha_2 \lambda_2^n U_2$ for $x = nL$ with $n \in \mathbb{Z}$, which asserts that there is no bounded solution of such frequencies k and these k determine photonic bandgap structures.

In fact, Floquet-Bloch theory reveals that σ_c forms the continuous spectrum of the problem (2.1), which is a union of finite intervals. Since $\mathrm{Tr}(P_L(k))$ is an analytic function of k, solutions to $\mathrm{Tr}(P_L(k)) = \pm 2$ are discrete and determine boundaries of intervals of the bandgap and continuous spectrum, i.e., $\sigma_c = \cup_{n=1}^{\infty} [(\gamma_n^-)^2, (\gamma_n^+)^2]$ with $\gamma_1^- = 0$ and $\gamma_n^- < \gamma_n^+ < \gamma_{n+1}^-$, where $|\mathrm{Tr}(P_L(\gamma_n^{\pm}))| = 2$, and the bandgap $\mathcal{B}$ is given by $\mathcal{B} = \cup_{n=1}^{\infty} ((\gamma_n^+)^2, (\gamma_{n+1}^-)^2)$.

Now, we consider a defected photonic crystal by replacing ε with ε_d defined by $\varepsilon_d(x) = \varepsilon(x)$ for $x < 0$, $\varepsilon_d(x) = \varepsilon_D$ for $0 < x < D$ and $\varepsilon_d(x) = \varepsilon(x - D)$ for $x > D$, i.e., the unit cells of the length L are repeatedly arranged on both sides of the defect of the permittivity ε_D of the thickness D. By P_D we denote the propagation matrix for the defect layer with $\mu_D = k\sqrt{\varepsilon_D} D$. The continuous spectrum σ_c of the perfect photonic crystal is not changed by local perturbations and the defected photonic crystal has the same continuous spectrum σ_c as the perfect one [2]. As opposed to the perfect photontic crystal, the defected one has localized defect modes whose frequencies are in the bandgap. To show this, suppose that there exists k in the bandgap satisfying

$$P_D U_1 = \alpha U_2 \tag{2.3}$$

for some constant α. Then Lemma 2.1 shows that the wave field vector of the solution with $U(0) = U_1$ satisfies

$$U(x+nL) = \lambda_2^n U(x) \text{ for } D \le x \le D + L,$$
$$U(x-nL) = \lambda_1^{-n} U(x) \text{ for } -L \le x \le 0$$

for $n = 0, 1, \dots$, which yields that the first component u of U is an exponentially decaying eigenmode and the corresponding k is a desired resonance frequency. If we denote the discrete set of frequencies of localized modes satisfying (2.3) in the bandgap by σ_p, then the spectrum σ_d of the defected photonic crystal can be written as $\sigma_d = \sigma_c \cup \sigma_p$.

3. Spectrum of supercell method

In this section, we analyze spectrum induced by the supercell method. When the supercell method with N periodic unit cells on both sides of the defect is considered, the photonic crystal again has a periodicity of period $2NL + D$. Since the coefficient of the differential equation is periodic, it has only a continuous spectrum as discussed for two layered media and let σ_N be the contiuous spectrum for the supercell method, which is also a disjoint union of closed intervals. The propagation matrix P_N for the supercell of size N is defined by

$$P_N = P_L^N P_D P_L^N$$

and hence the continuous spectrum σ_N is determined by the inequality $|\mathrm{Tr}(P_N(k))| \leq 2$. For k such that $|\mathrm{Tr}(P_L(k))| \neq 2$, the propagation matrix P_L is diagonalized with respect to eigenvectors U_1 and U_2, i.e.,

$$M^{-1} P_L M = \Lambda,$$

where Λ is the diagonal matrix whose diagonal entries are λ_1 and λ_2 and M is the invertible matrix whose columns consist of U_1 and U_2. By using the characteristic polynomial P_L of λ, here we choose explicitly the eigenvectors

$$U_1 = \begin{bmatrix} \lambda_1 - p_{22} \\ p_{21} \end{bmatrix}, \quad U_2 = \begin{bmatrix} \lambda_2 - p_{22} \\ p_{21} \end{bmatrix}.$$

A simple computation yields that

$$|\mathrm{Tr}(P_N)| = |\mathrm{Tr}(P_D (P_L)^{2N})| = |\mathrm{Tr}(P_D M \Lambda^{2N} M^{-1})| = |\mathrm{Tr}(M^{-1} P_D M \Lambda^{2N})|.$$

Let $T = M^{-1} P_D M$ and t_{ij} denote the (i, j) components of T, in other words,

$$P_D U_1 = t_{11} U_1 + t_{21} U_2 \quad \text{and} \quad P_D U_2 = t_{12} U_1 + t_{22} U_2. \tag{3.1}$$

Therefore, $k^2 \in \sigma_p$ if and only if $t_{11}(k) = 0$. Here we note that $\lambda_1, \lambda_2, U_1, U_2$ and t_{ij} are analytic in a neighborhood of k satisfying $|\mathrm{Tr}(P_L(k))| \neq 2$ as pointed out in [6]. Clearly, we have

$$\mathrm{Tr}(P_N) = t_{11} \lambda_1^{2N} + t_{22} \lambda_2^{2N}. \tag{3.2}$$

Lemma 3.1. *For $k^2 \in \sigma_c$ such that $|\mathrm{Tr}(P_L(k))| < 2$, it holds that $|t_{11}| \geq 1$ and*

$$|\mathrm{Tr}(P_N(k))| = |2\Re(t_{11} e^{2iN\tau})|.$$

Proof. Noting that U_1 and U_2 are complex conjugates of each other, taking the complex conjugate of (3.1) yields that

$$P_D \bar{U}_1 = \bar{t}_{11} \bar{U}_1 + \bar{t}_{21} \bar{U}_2 = \bar{t}_{11} U_2 + \bar{t}_{21} U_1$$

equals

$$P_D U_2 = t_{12} U_1 + t_{22} U_2,$$

which implies that $t_{22} = \bar{t}_{11}$ and $t_{12} = \bar{t}_{21}$. Consequently, since $\det(T) = |t_{11}|^2 - |t_{12}|^2 = 1$, it is obvious that $|t_{11}| \geq 1$. Also, from (3.2)

$$|\mathrm{Tr}(P_N(k))| = |t_{11} e^{2iN\tau} + \bar{t}_{11} e^{-2iN\tau}| = |2\Re(t_{11} e^{2iN\tau})|,$$

which completes the proof. $\square$

Let θ_{11} be the argument of t_{11}. Lemma 3.1 shows that $|\mathrm{Tr}(P_N)| = 2|t_{11}||\cos(2N\tau + \theta_{11})|$ on the continuous spectrum σ_c of the perfect photonic crystal is not guaranteed to be bounded by 2 as $|t_{11}| \geq 1$, but it is highly oscillatory between $\pm 2|t_{11}|$ for large N. However we can show the convergence of σ_N to σ_c in the sense of the following theorem.

Theorem 3.2. *Let $k_0^2 \in \sigma_c$. Then for any $\varepsilon > 0$ there exists N_0 such that for any integer $N > N_0$ there is $k^2 \in \sigma_N$ satisfying $|k - k_0| < \varepsilon$.*

Proof. We first consider the case $k_0^2 \in \sigma_c$ such that $|\mathrm{Tr}(P_L(k_0))| < 2$. In this case, since λ_1 is analytic near k_0, τ is a non-constant continuous function of k in a ε_1-neighborhood $B(k_0, \varepsilon_1)$ of k_0 for some $0 < \varepsilon_1 \leq \varepsilon$ and hence an interval is included in $\tau(B(k_0, \varepsilon_1))$, that is, there exist constants $\tau_m < \tau_M$ such that $(\tau_m, \tau_M) \subset \tau(B(k_0, \varepsilon_1))$. Now, we choose $N_0 = \pi/(\tau_M - \tau_m)$. Then since $\tau_M - \tau_m$ is larger than the period of $\cos(2N\tau + \theta_{11})$ (as a function of τ) for any integer $N > N_0$, there exists $k \in B(k_0, \varepsilon_1)$ such that $|\mathrm{Tr}(P_N)| = 0$, which implies that $k^2 \in \sigma_N$ and $|k - k_0| < \varepsilon$.

The assertion for the case $k_0^2 \in \sigma_c$ with $|\mathrm{Tr}(P_N(k_0))| = 2$ also holds since k_0^2 is a limit point of a sequence $k_n^2 \in \sigma_c$ with $|\mathrm{Tr}(P_N(k_n))| < 2$. $\square$

The next theorem is concerned with the behavior of $\mathrm{Tr}(P_N(k))$ for k^2 in the bandgap $\mathcal{B}$.

Theorem 3.3. *For any compact set $K \subset \mathcal{B} \setminus \sigma_p$, then*

$$|\mathrm{Tr}(P_N(k))| \to \infty \ \textit{as}\ N \to \infty \tag{3.3}$$

uniformly for $k^2 \in K$. For $k^2 \in \sigma_p$,

$$|\mathrm{Tr}(P_N(k))| \to 0 \quad \textit{as}\ N \to \infty. \tag{3.4}$$

Proof. Since $P_D(k)U_1$ is not parallel to U_2 for $k^2 \in K$ and t_{11} is analytic, $|t_{11}|$ is bounded below away from zero in K due to (3.1), saying $|t_{11}| > t_m$ in K for some positive constant t_m. Also, there exist constants t_M and λ_m such that $|t_{22}| < t_M$ and $1 < |\lambda_m| < |\lambda_1|$ in K by analyticity of t_{22} and λ_1. Hence we have

$$|\mathrm{Tr}(P_N)| > |t_m||\lambda_m|^{2N} - \frac{|t_M|}{|\lambda_m|^{2N}}$$

by (3.2), which implies that $|\mathrm{Tr}(P_N)|$ tends toward infinity uniformly in K as N approaches infinity.

On the other hands, if $k^2 \in \sigma_p$, then $P_D U_1$ is parallel to U_2 and hence we have $t_{11} = 0$ in (3.1), from which (3.4) immediately follows from (3.2) since $|\lambda_2| < 1$. $\square$

The convergence result of (3.4) shows that each point spectrum of the defected photonic crystal is in the continuous spectrum of the supercell method for large N. The following lemma proved in [6] is required for the asymptotic behavior of the continuous spectrum of the supercell method near $k_p^2 \in \sigma_p$.

Lemma 3.4. *For $k_p^2 \in \sigma_p$, it holds that*

$$\det(U_1, U_1') < 0 \quad \textit{and} \quad \det(U_2, U_2') > 0$$

at $k = k_p$, where $'$ represents differentiation with respect to k.

Lemma 3.5. *For $k_p^2 \in \sigma_p$, it holds that $t_{11}'(k_p) \neq 0$.*

Proof. Since $t_{11} = \det(P_D U_1, U_2)/\det(U_1, U_2)$ and $\det(P_D U_1, U_2)(k_p) = 0$, it suffices to show that

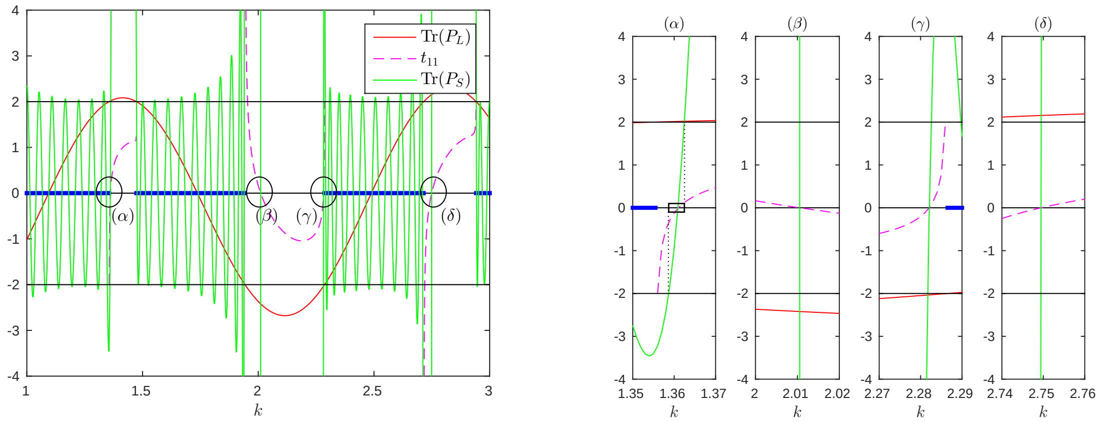
$$(\det(P_D U_1, U_2))'(k_p) \neq 0. \tag{3.5}$$

By invoking that $P_D U_1 = \alpha U_2$ for some $\alpha \neq 0$ at $k = k_p$ and $\det(P_D) = 1$, it can be shown that at $k = k_p$

$$\begin{aligned}(\det(P_D U_1, U_2))' &= \det(P_D' U_1, U_2) + \det(P_D U_1', U_2) + \det(P_D U_1, U_2') \\ &= \frac{1}{\alpha}\Big[\det(P_D' U_1, P_D U_1) + (\det(U_1', U_1) + \alpha^2 \det(U_2, U_2'))\Big].\end{aligned}$$

A simple computation for the first term leads to

$$\det(P_D' U_1, P_D U_1) = A(U_1)_1^2 + 2B(U_1)_1(U_1)_2 + C(U_1)_2^2$$



(a) (b)

Figure 1: Graphs of $\mathrm{Tr}(P_L)$, t_{11} and $\mathrm{Tr}(P_N)$ for $N = 10$.

with

$$A = \frac{1}{2\sqrt{\varepsilon_D}}(2\mu_D + \sin(2\mu_D)), \quad B = \frac{1}{k}\sin^2(\mu_D), \quad C = \frac{\sqrt{\varepsilon_D}}{2k^2}(2\mu_D - \sin(2\mu_D)).$$

Since $A, C \geq 0$ and $B^2 - AC = (-\mu_D^2 + \sin^2(\mu_D))/k^2 \leq 0$, we have $\det(P_D'U_1, P_DU_1) \geq 0$. Finally, Lemma 3.4 shows

$$(\det(P_DU_1, U_2))' = \det(P_D'U_1, P_DU_1) - \det(U_1, U_1') + \alpha^2 \det(U_2, U_2') > 0,$$

which leads to (3.5). □

Now, we are in position to show that intervals of the continuous spectrum of the supercell method containing frequencies of localized defected modes shrink exponentially with increasing size of the supercell.

Theorem 3.6. *For $k_p^2 \in \sigma_p$, the length of the interval of σ_N containing k_p^2 decays exponentially, i.e., if $[k_1^2, k_2^2]$ is the interval of σ_N containing k_p^2, then $|k_2^2 - k_1^2| = O(\lambda_*^{-2N})$ for some $\lambda_* > 1$ as $N \to \infty$.*

Proof. By Lemma 3.5 there exists a δ-neighborhood $B(k_p, \delta)$ of k_p in the bandgap such that t_{jj} and λ_j are analytic, $t_{11}' \neq 0$ and $|\lambda_1| > \lambda_* > 1$ in $B(k_p, \delta)$ for some constant λ_*. By applying Taylor theorem to t_{11} with $t_{11}(k_p) = 0$, there exists ξ between k_p and $k \in B(k_p, \delta)$ satisfying

$$\mathrm{Tr}(P_N(k)) = t_{11}'(\xi)(k - k_p)\lambda_1^{2N} + t_{22}\lambda_2^{2N}. \tag{3.6}$$

Now, it holds that if $k^2 \in \sigma_N$ and $k \in B(k_p, \delta)$, then $|k - k_p| < C\lambda_*^{-2N}$ for a positive constant C independent of N. Indeed, for $k^2 \in \sigma_N$ with $k \in B(k_p, \delta)$, we have $|\mathrm{Tr}(P_N(k))| \leq 2$ and the equation (3.6), from which it follows that

$$|k - k_p| \leq \frac{2 + \sup_{k \in B(k_p,\delta)} |t_{22}\lambda_2^{2N}|}{\inf_{k \in B(k_p,\delta)} |t_{11}'\lambda_1^{2N}|} \leq C\lambda_*^{-2N}, \tag{3.7}$$

as t_{22}, λ_1 and λ_2 are analytic and $|t_{11}'|$ is bounded below away from zero in $B(k_p, \delta)$. Thus, for N large enough that $C\lambda_*^{-2N} < \delta$, the interval $[k_1^2, k_2^2]$ of σ_N containing k_p^2 is included in $B(k_p, \delta)$ and furthermore (3.7) yields $|k_2^2 - k_1^2| = O(\lambda_*^{-2N})$ as $N \to \infty$, which completes the proof. □

Finally, we present Figure 1 of the graphs of $\mathrm{Tr}(P_L)$, t_{11} and $\mathrm{Tr}(P_N)$ with $N = 10$ as functions of k to illustrate the theoretical results. Here the material constants are given by

$$\varepsilon_1 = 1, L_1 = 2, \quad \varepsilon_2 = 6, L_2 = 1, \quad \varepsilon_D = 2, L_D = 3.$$

The red solid curve is the graph of $\mathrm{Tr}(P_L)$ and the blue intervals are connected components of the continuous spectrum σ_c of the defected photonic crystal. The graph of t_{11} is represented by the dashed magenta curve and its zeros in the circles $(\alpha), (\beta), (\gamma)$ and (δ) are frequencies of defected eigenmodes in σ_p. The green curve is the graph of $\mathrm{Tr}(P_N)$. As studied, the high oscillation of $\mathrm{Tr}(P_N)$ on σ_c and steep slopes near the zeros of t_{11} in the bandgap can be observed in Figure 1 (a) and (b). The black rectangle in Figure 1 (b)-(α) signifies an interval of σ_N containing a defect frequency k_p^2 and its length decays exponentially with increasing N.

[1] A. Figotin and V. Gorentsveig. Localized electromagnetic waves in a layered periodic dielectric medium with a defect. *Phys. Rev. B*, 58:180–188, 1998.

[2] A. Figotin and A. Klein. Localized classical waves created by defects. *Journal of Statistical Physics*, 86(1-2):165–177, 1997.

[3] R. A. Horn and C. R. Johnson. *Topics in matrix analysis*. Cambridge University Press, Cambridge, 1991.

[4] J. D. Joannopoulos, S. G. Johnson, J. N. Winn, and R. D. Meade. *Photonic Crystals: Molding the Flow of Light*. Princeton University Press, New Jersey, 2008.

[5] S. Kim. Cartesian PML approximation to resonances in open systems in $\mathbb{R}^2$. *Appl. Numer. Math.*, 81:50–75, 2014.

[6] J. Lin and F. Santosa. Resonances of a finite one-dimensional photonic crystal with a defect. *SIAM J. Appl. Math.*, 73(2):1002–1019, 2013.

[7] S. Soussi. Convergence of the supercell method for defect modes calculations in photonic crystals. *SIAM J. Numer. Anal.*, 43(3):1175–1201, 2005.