FRACTIONAL ORDER SOBOLEV SPACES FOR THE NEUMANN LAPLACIAN AND THE VECTOR LAPLACIAN

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Abstract. In this paper we study fractional Sobolev spaces characterized by a norm based on eigenfunction expansions. The goal of this paper is twofold. The first one is to define fractional Sobolev spaces of order $-1 \leq s \leq 2$ equipped with a norm defined in terms of Neumann eigenfunction expansions. Due to the zero Neumann trace of Neumann eigenfunctions on a boundary, fractional Sobolev spaces of order $3/2 \leq s \leq 2$ characterized by the norm are the spaces of functions with zero Neumann trace on a boundary. The spaces equipped with the norm are useful for studying cross-sectional traces of solutions to the Helmholtz equation in waveguides with a homogeneous Neumann boundary condition. The second one is to define fractional Sobolev spaces of order $-1 \leq s \leq 1$ for vector-valued functions in a simply-connected, bounded and smooth domain in $\mathbb{R}^2$. These spaces are defined by a norm based on series expansions in terms of eigenfunctions of the vector Laplacian with boundary conditions of zero tangential component or zero normal component. The spaces defined by the norm are important for analyzing cross-sectional traces of time-harmonic electromagnetic fields in perfectly conducting waveguides.

1. Introduction

This paper deals with fractional Sobolev spaces characterized by a norm based on eigenfunction expansions associated with the scalar Laplacian and the vector Laplacian on bounded and smooth domains. There are many ways to define a norm in fractional Sobolev spaces, which is equivalent to each other, such as Fourier transformation, Slobodeckij semi-norm or interpolation method [17]. Among others, the formula of the norm presented in this paper is useful for studying a fractional Laplace operator [3] and, in particular, for series representations of solutions to the Helmholtz equation and the Maxwell's equations posed in waveguides because their traces on cross-sections of waveguides can be written as series expansions in terms of cross-sectional eigenfunctions of the scalar Laplacian [2, 13, 14] and the vector Laplacian [1, 12]. They are also utilized importantly to define the Dirichlet-to-Neumann operator [11, 18] and the Electric-to-Magnetic operator [12] crucial to understand radiating solutions for wave propagation problems.

Let Ω be a bounded and smooth domain in $\mathbb{R}^d$, $d = 2$ or 3. We use usual notations for Sobolev spaces, for example, $L^2(\Omega)$ is the set of square integrable functions on Ω and $H^k(\Omega)$ for k positive integer is the subspace of $L^2(\Omega)$ of functions whose derivatives up to the k-th order are square integrable as well, with $H^0(\Omega) = L^2(\Omega)$. We denote the L^2-inner product on Ω by $(\cdot, \cdot)_\Omega$ and the H^1-inner product by $(\cdot, \cdot)_{1,\Omega}$. In addition, $\widetilde{H}^{-1}(\Omega)$ represents the dual space of $H^1(\Omega)$ with the pivot space $L^2(\Omega)$,

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and $\langle\cdot,\cdot\rangle_{1,\Omega}$ denotes the duality pairing between $\widetilde{H}^{-1}(\Omega)$ and $H^1(\Omega)$. Fractional Sobolev spaces $H^s(\Omega)$ of order $0 < s < 1$ can be defined by the real interpolation [5, 15]

$$H^s(\Omega) = [H^0(\Omega),\ H^1(\Omega)]_s$$

and their norms are defined by

$$\|u\|_{H^s(\Omega)} = C_s \left(\int_0^\infty K(u,t,H^0(\Omega),H^1(\Omega))^2 t^{-2s-1} dt \right)^{1/2} \tag{1.1}$$

with $C_s = \sqrt{2\sin(\pi s)/\pi}$ and

$$K(u,t,H^0(\Omega),H^1(\Omega)) = \inf_{\theta\in H^1(\Omega)} (\|u-\theta\|^2_{H^0(\Omega)} + t^2\|\theta\|^2_{H^1(\Omega)})^{1/2}.$$

Also, $\widetilde{H}^s(\Omega)$ for $-1 < s < 0$ and $H^s(\Omega)$ for $1 < s < 2$ are defined as

$$H^s(\Omega) = [H^1(\Omega),\ H^2(\Omega)]_{s-1}, \qquad 1 < s < 2,$$
$$\widetilde{H}^s(\Omega) = [\widetilde{H}^{-1}(\Omega),\ H^0(\Omega)]_{s+1}, \quad -1 < s < 0$$

with norms defined analogously to (1.1). For simple presentation, let $\mathcal{H}^s(\Omega) = H^s(\Omega)$ for $s \geq 0$ with $\mathcal{H}^0(\Omega) = L^2(\Omega)$ and $\mathcal{H}^s(\Omega) = \widetilde{H}^s(\Omega)$ for $s < 0$.

It is worth beginning with a review on a result in [3] related to the main goal of this paper but for Dirichlet boundary value problems. For Dirichlet boundary value problems, the interpolation space

$$\mathbb{H}^s(\Omega) := \begin{cases} H^1_0(\Omega)\cap H^s(\Omega), & 1\le s\le 2,\\ [L^2(\Omega),\ H^1_0(\Omega)]_s, & 0\le s\le 1,\\ [H^{-1}(\Omega),\ L^2(\Omega)]_{1+s}, & -1\le s\le 0 \end{cases}$$

is importantly used for regularity estimates. Here $H^1_0(\Omega)$ is the subspace in $H^1(\Omega)$ of functions with zero Dirichlet trace on $\partial\Omega$ and $H^{-1}(\Omega)$ is the dual space of $H^1_0(\Omega)$. In [3], it is shown that for a complete orthonormal basis $\{V_n\}_{n=1}^\infty$ consisting of Dirichlet eigenfunctions associated with eigenvalues μ_n, the interpolation space is identical with a space defined in terms of a norm based on Dirichlet eigenfunction expansions. More precisely, the space $\dot{H}^s(\Omega)$ of functions $u = \sum_{n=1}^\infty u_n V_n$ satisfying

$$\|u\|_{\dot{H}^s(\Omega)} := \left(\sum_{n=1}^\infty (1+\mu_n)^s |u_n|^2 \right)^{1/2} < \infty \tag{1.2}$$

coincides with the interpolation space $\mathbb{H}^s(\Omega)$ and their norms are equivalent.

The aim of this paper is twofold and it is a development of similar results for the Neumann Laplacian and the vector Laplacian. The first main result for the Neumann Laplacian is as follows: let $\{Y_n\}_{n=0}^\infty$ be a complete orthonormal basis consisting of Neumann eigenfunctions pertaining to eigenvalues λ_n. We introduce a space $\dot{\mathcal{H}}^s(\Omega)$ of functions $u = \sum_{n=0}^\infty u_n Y_n$ satisfying

$$\|u\|_{\dot{\mathcal{H}}^s(\Omega)} := \left(\sum_{n=0}^\infty (1+\lambda_n)^s |u_n|^2 \right)^{1/2} < \infty \tag{1.3}$$

for $-1 \le s \le 2$. We will show that for $-1 \le s \le 1$ the space $\dot{\mathcal{H}}^s(\Omega)$ is identical with the interpolation space $\mathcal{H}^s(\Omega)$, and the norm (1.3) of $\dot{\mathcal{H}}^s(\Omega)$ coincides with the norm (1.1) (analogously defined for $-1 < s < 0$). In case of $1 < s \le 2$, the analysis for $\dot{\mathcal{H}}^s(\Omega)$ is more involved, since Y_n has zero Neumann trace on $\partial\Omega$, $\{Y_n\}_{n=0}^\infty$ is not dense in $\mathcal{H}^s(\Omega)$ for $3/2 \leq s \leq 2$ and hence $\dot{\mathcal{H}}^s(\Omega)$ is a proper subspace of $\mathcal{H}^s(\Omega)$. In this case, we restrict $\mathcal{H}^s(\Omega)$ to a subspace of functions with zero Neumann trace on $\partial\Omega$, then it turns out that two spaces are identical and the norms are equivalent. More precisely, let $\mathcal{H}_n^2(\Omega)$ be a subspace of functions with zero Neumann trace on $\partial\Omega$ in $\mathcal{H}^2(\Omega)$ and define an interpolation space $\mathcal{H}_n^s(\Omega) := [\mathcal{H}^1(\Omega),\ \mathcal{H}_n^2(\Omega)]_{s-1}$ for $1 \leq s \leq 2$. Then we can show that two spaces $\mathcal{H}_n^s(\Omega)$ and $\dot{\mathcal{H}}^s(\Omega)$ for $1 \leq s \leq 2$ coincide and the norms are equivalent. The space $\mathcal{H}_n^s(\Omega)$ is well-analyzed in [9, 16], showing that $\mathcal{H}_n^s(\Omega) = \mathcal{H}^s(\Omega)$ for $1 \leq s < 3/2$ (and so $\mathcal{H}^s(\Omega) = \dot{\mathcal{H}}^s(\Omega)$) and $\mathcal{H}_n^s(\Omega) = \{u \in \mathcal{H}^s(\Omega) \ : \ \partial u/\partial \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega\}$ for $3/2 < s \leq 2$. For $s = 3/2$, functions in $\mathcal{H}_n^{3/2}(\Omega)$ has a Neumann trace which vanishes on $\partial\Omega$ in a special sense.

The second part is devoted to studying fractional Sobolev spaces of order $-1 \leq s \leq 1$ consisting of vector-valued functions related to the boundary conditions of zero tangential component or zero normal component in a simply-connected, bounded and smooth domain $\Omega \subset \mathbb{R}^2$. These spaces can be defined by a norm based on eigenfunction expansions for the vector Laplacian supplemented with zero tangential component or zero normal component on $\partial\Omega$ for the essential boundary condition. It can be found in [1, 12] that they play an important role for an analysis of time-harmonic electromagnetic wave propagation in perfectly conducting waveguides in $\mathbb{R}^3$.

The remaining part of the paper is composed of two sections. In section 2 we will analyze fractional Sobolev spaces $\dot{\mathcal{H}}^s(\Omega)$ for $-1 \leq s \leq 2$ based on Neumann eigenfunction expansions. It will be shown that they coincide with $\mathcal{H}^s(\Omega)$ for $-1 \leq s \leq 1$ and $\mathcal{H}_n^s(\Omega)$ for $1 \leq s \leq 2$ and their norms are equivalent. The result is established thanks to a standard spectral theory [19] of a compact self-adjoint operator and the real interpolation technique [5, 15]. In section 3 we will study fractional Sobolev spaces of order $-1 \leq s \leq 1$ for vector-valued functions in Ω. Here we start with eigenvalue problems of the vector Laplacian and follow the same lines as those in section 2 to obtain a characterization of the spaces and equivalent norms based on eigenfunction expansions. In each section, we provide an analysis for cross-sectional trace operators in waveguides as an application of fractional Sobolev spaces equipped with the norms based on eigenfunction expansions.

2. Fractional Sobolev spaces for Neumann boundary value problems

In this section we will define fractional Sobolev spaces of order $-1 \leq s \leq 2$ associated with Neumann boundary value problems in a bounded and smooth domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3. We first consider the Neumann Laplacian in the weak sense and define a solution operator pertaining to the Neumann Laplacian. Since the solution operator is continuous, compact and self-adjoint, the spectral theory [19] comes into play for series expansions of functions in terms of Neumann eigenfunctions. The main idea of this section is one used in [3]. Some of the analysis are somewhat elementary but we will provide them for completeness.

2.1. **Preliminaries.**

We first introduce $\mathcal{L} : \mathcal{H}^1(\Omega) \to \mathcal{H}^{-1}(\Omega)$ associated with the Neumann Laplace operator defined by

$$\langle \mathcal{L}(u), v \rangle_{1,\Omega} := (\nabla u, \nabla v)_\Omega + (u, v)_\Omega = (u, v)_{1,\Omega} \quad \text{for all } u, v \in \mathcal{H}^1(\Omega),$$

which is continuous from $\mathcal{H}^1(\Omega)$ to $\mathcal{H}^{-1}(\Omega)$ and satisfies $\|\mathcal{L}(u)\|_{\mathcal{H}^{-1}(\Omega)} = \|u\|_{\mathcal{H}^1(\Omega)}$. The dual space $\mathcal{H}^{-1}(\Omega)$ of $\mathcal{H}^1(\Omega)$ is equipped with the standard operator norm

$$\|F\|_{\mathcal{H}^{-1}(\Omega)} := \sup_{0\neq\phi\in\mathcal{H}^1(\Omega)} \frac{\langle F,\phi\rangle_{1,\Omega}}{\|\phi\|_{\mathcal{H}^1(\Omega)}} \tag{2.1}$$

for $F \in \mathcal{H}^{-1}(\Omega)$. Also, due to the Lax-Milgram lemma, we can define the solution operator

$$\mathcal{T} : \mathcal{H}^{-1}(\Omega) \to \mathcal{H}^1(\Omega) \subset \mathcal{H}^{-1}(\Omega)$$

by $\mathcal{T}(F)$ for $F \in \mathcal{H}^{-1}(\Omega)$ satisfying

$$(\mathcal{T}(F), v)_{1,\Omega} = \langle F, v\rangle_{1,\Omega} \quad \text{for all } v \in \mathcal{H}^1(\Omega). \tag{2.2}$$

It is obvious that $\mathcal{L}\mathcal{T} = I_{\mathcal{H}^{-1}(\Omega)}$ and $\mathcal{T}\mathcal{L} = I_{\mathcal{H}^1(\Omega)}$.

Now, by using the solution operator $\mathcal{T}$, we can define the inner product $(\cdot,\cdot)_{-1,\Omega}$ in $\mathcal{H}^{-1}(\Omega)$ by $(F,G)_{-1,\Omega} := \langle F, \mathcal{T}(G)\rangle_{1,\Omega}$ for $F, G \in \mathcal{H}^{-1}(\Omega)$. By the definition of the operator $\mathcal{T}$, it holds that

$$(F,G)_{-1,\Omega} = (\mathcal{T}(F), \mathcal{T}(G))_{1,\Omega}. \tag{2.3}$$

The norm of $\mathcal{H}^{-1}(\Omega)$ induced from the inner product $(\cdot,\cdot)_{-1,\Omega}$ coincides with the operator norm (2.1),

$$\|F\|_{\mathcal{H}^{-1}(\Omega)} = (F,F)^{1/2}_{-1,\Omega}.$$

2.2. **Orthonormal bases of Neumann eigenfunctions.** We consider the Neumann eigenvalue problem of $-\Delta$,

$$\begin{aligned} -\Delta Y &= \lambda Y \quad \text{in } \Omega, \\ \frac{\partial Y}{\partial \boldsymbol{\nu}} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.4}$$

where $\boldsymbol{\nu}$ stands for the outward unit normal vector on $\partial\Omega$. It is well-known, e.g. in [6, 7], that there exist non-negative real eigenvalues λ_n such that

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

and $\lambda_n \to \infty$ as $n \to \infty$, and eigenfunctions $Y_n \in L^2(\Omega)$ associated with λ_n, which form an orthonormal basis in $L^2(\Omega)$.

In the sequel, we discuss complete orthonormal bases consisting of eigenfunctions of $\mathcal{T}$ as an operator defined in three different spaces $\mathcal{H}^{-1}(\Omega)$, $\mathcal{H}^0(\Omega)$ and $\mathcal{H}^1(\Omega)$.

Lemma 2.1. *The operator $\mathcal{T}$ is a continuous and compact operator from $\mathcal{H}^{-1}(\Omega)$ to $\mathcal{H}^{-1}(\Omega)$ satisfying*

$$\|\mathcal{T}(F)\|_{\mathcal{H}^{-1}(\Omega)} \leq \|F\|_{\mathcal{H}^{-1}(\Omega)}.$$

Proof. We use the continuous embedding of $\mathcal{H}^1(\Omega)$ into $\mathcal{H}^{-1}(\Omega)$ and (2.3) to obtain

$$\|\mathcal{T}(F)\|_{\mathcal{H}^{-1}(\Omega)} \leq \|\mathcal{T}(F)\|_{\mathcal{H}^1(\Omega)} = \|F\|_{\mathcal{H}^{-1}(\Omega)}.$$

Also, since $\mathcal{H}^1(\Omega)$ is compactly embedded in $\mathcal{H}^{-1}(\Omega)$, $\mathcal{T} : \mathcal{H}^{-1}(\Omega) \to \mathcal{H}^{-1}(\Omega)$ is compact. □

Lemma 2.2. *The linear map $\mathcal{T} : \mathcal{H}^{-1}(\Omega) \to \mathcal{H}^{-1}(\Omega)$ is self-adjoint with respect to $(\cdot,\cdot)_{-1,\Omega}$.*

Proof. The proof proceeds with repeated use of definitions of the inner product in $\mathcal{H}^{-1}(\Omega)$ and the operator $\mathcal{T}$ as follows: for $F, G \in \mathcal{H}^{-1}(\Omega)$,

$$\begin{aligned}(\mathcal{T}(F), G)_{-1,\Omega} &= \langle \mathcal{T}(F), \mathcal{T}(G)\rangle_{1,\Omega} = (\mathcal{T}(F), \mathcal{T}(G))_\Omega \\ &= (\mathcal{T}(G), \mathcal{T}(F))_\Omega = \langle \mathcal{T}(G), \mathcal{T}(F)\rangle_{1,\Omega} \\ &= (\mathcal{T}(G), F)_{-1,\Omega} = (F, \mathcal{T}(G))_{-1,\Omega},\end{aligned}$$

which shows that $\mathcal{T}$ is self-adjoint. □

Lemma 2.3. *There exists an orthonormal basis $\{\widetilde{Y}_n\}_{n=0}^{\infty}$ with respect to $(\cdot,\cdot)_{-1,\Omega}$ consisting of eigenfunctions of $\mathcal{T} : \mathcal{H}^{-1}(\Omega) \to \mathcal{H}^{-1}(\Omega)$ associated with eigenvalues $\eta_n := (1+\lambda_n)^{-1}$, that is, $\mathcal{T}(\widetilde{Y}_n) = (1+\lambda_n)^{-1}\widetilde{Y}_n$.*

Proof. The existence of an orthonormal basis consisting of eigenfunctions $\widetilde{Y}_n$ of $\mathcal{T}$ in $\mathcal{H}^{-1}(\Omega)$ is established by the Hilbert-Schmidt theorem in the spectral theory [19] as $\mathcal{T}$ is continuous, compact and self-adjoint in $\mathcal{H}^{-1}(\Omega)$ proved in Lemma 2.1 and Lemma 2.2.

To show that eigenvalues η_n of $\mathcal{T}$ are of the form $\eta_n = (1+\lambda_n)^{-1}$, let $\widetilde{Y}_n$ be an eigenfunction for an eigenvalue η_n. Then it holds that

$$(\mathcal{T}(\widetilde{Y}_n), v)_{1,\Omega} = \langle \widetilde{Y}_n, v\rangle_{1,\Omega} \quad \text{for all } v \in \mathcal{H}^1(\Omega)$$

by the definition of $\mathcal{T}$. From the fact that $\mathcal{T}(\widetilde{Y}_n) = \eta_n \widetilde{Y}_n \in \mathcal{H}^1(\Omega)$ it follows that

$$\eta_n(\widetilde{Y}_n, v)_{1,\Omega} = (\widetilde{Y}_n, v)_\Omega \quad \text{for all } v \in \mathcal{H}^1(\Omega).$$

Now it can be written as $(\nabla\widetilde{Y}_n, \nabla v)_\Omega = (\eta_n^{-1} - 1)(\widetilde{Y}_n, v)_\Omega$ for all $v \in \mathcal{H}^1(\Omega)$, which reveals that η_n satisfies $\lambda_n = \eta_n^{-1} - 1$ for an eigenvalue λ_n associated for the Neumann eigenfunction $\widetilde{Y}_n$. □

Lemma 2.4. *Let $Y_n = (1+\lambda_n)^{-1/2}\widetilde{Y}_n$. Then $\{Y_n\}_{n=0}^{\infty}$ is a complete orthonormal basis of $\mathcal{H}^0(\Omega)$ with respect to the L^2-inner product $(\cdot,\cdot)_\Omega$ consisting of eigenfunctions of $\mathcal{T} : \mathcal{H}^0(\Omega) \to \mathcal{H}^0(\Omega)$.*

Proof. We first observe that

$$\begin{aligned}\delta_{m,n} &= (\widetilde{Y}_m, \widetilde{Y}_n)_{-1,\Omega} = \langle \widetilde{Y}_m, \mathcal{T}(\widetilde{Y}_n)\rangle_{1,\Omega} \\ &= \langle \widetilde{Y}_m, (1+\lambda_n)^{-1}\widetilde{Y}_n\rangle_{1,\Omega} = (1+\lambda_n)^{-1}(\widetilde{Y}_m, \widetilde{Y}_n)_\Omega,\end{aligned}$$

from which it follows that $\widetilde{Y}_n$ is orthogonal with respect to the inner product $(\cdot,\cdot)_\Omega$ and $\|\widetilde{Y}_n\|_{\mathcal{H}^0(\Omega)} = (1+\lambda_n)^{1/2}$. Since Y_n is an eigenfunction in $\mathcal{H}^0(\Omega)$ of $\mathcal{T}$ and $\{Y_n\}_{n=0}^{\infty}$ has all eigenfunctions of the continuous, compact and self-adjoint operator $\mathcal{T} : \mathcal{H}^0(\Omega) \to \mathcal{H}^0(\Omega)$, $\{Y_n\}_{n=0}^{\infty}$ is a complete orthonormal basis of $\mathcal{H}^0(\Omega)$. □

Lemma 2.5. *Let $\hat{Y}_n = (1+\lambda_n)^{-1/2}Y_n$. Then $\{\hat{Y}_n\}_{n=0}^{\infty}$ is a complete orthonormal basis of $\mathcal{H}^1(\Omega)$ with respect to the $\mathcal{H}^1$-inner product $(\cdot,\cdot)_{1,\Omega}$ consisting of eigenfunctions of $\mathcal{T} : \mathcal{H}^1(\Omega) \to \mathcal{H}^1(\Omega)$.*

Proof. It is clear that $\hat{Y}_n$ are eigenfunctions of $\mathcal{T}$ in $\mathcal{H}^1(\Omega)$. Also, $\hat{Y}_n$ are orthonormal with respect to $(\cdot,\cdot)_{1,\Omega}$. In fact, by utilizing (2.3) we are led to

$$(\hat{Y}_m, \hat{Y}_n)_{1,\Omega} = (\mathcal{T}(\widetilde{Y}_m), \mathcal{T}(\widetilde{Y}_n))_{1,\Omega} = (\widetilde{Y}_m, \widetilde{Y}_n)_{-1,\Omega} = \delta_{m,n}.$$

For completeness of $\{\hat{Y}_n\}_{n=0}^{\infty}$ in $\mathcal{H}^1(\Omega)$, we choose any $u \in \mathcal{H}^1(\Omega)$ and suppose that

$$(\hat{Y}_n, u)_{1,\Omega} = 0 \quad \text{for all } n = 0, 1, 2, \ldots.$$

Then by the definition of $\mathcal{T}$, we have

$$\begin{aligned} 0 = (\hat{Y}_n, u)_{1,\Omega} &= (1+\lambda_n)(\mathcal{T}(\hat{Y}_n), u)_{1,\Omega} = (1+\lambda_n)\langle \hat{Y}_n, u\rangle_{1,\Omega} \\ &= (1+\lambda_n)(\hat{Y}_n, u)_{\Omega} = (1+\lambda_n)^{1/2}(Y_n, u)_{\Omega}. \end{aligned}$$

Since $\{Y_n\}_{n=0}^{\infty}$ is dense in $\mathcal{H}^0(\Omega)$, it follows that $u = 0$, which establishes the completeness of $\{Y_n\}_{n=0}^{\infty}$ in $\mathcal{H}^1(\Omega)$. □

2.3. **Fractional Sobolev spaces $\dot{\mathcal{H}}^s(\Omega)$ of order $-1 \le s \le 1$.** In this subsection, we study fractional Sobolev spaces $\dot{\mathcal{H}}^s(\Omega)$ of order $-1 \le s \le 1$ characterized by a norm based on series expansions in terms of Neumann eigenfunctions in $\mathcal{H}^0(\Omega)$. We recall

$$\dot{\mathcal{H}}^s(\Omega) = \{u = \sum_{n=0}^{\infty} u_n Y_n \ : \ \|u\|_{\dot{\mathcal{H}}^s(\Omega)} < \infty\}$$

with the definition (1.3) of $\|\cdot\|_{\dot{\mathcal{H}}^s(\Omega)}$. We note that fractional Sobolev spaces $\dot{\mathcal{H}}^s(\Omega)$ of order $-1 \le s \le 2$ can be interpreted as interpolation spaces. The following lemma can be proved as in [5, Appendix B].

Lemma 2.6. *The fractional Sobolev space $\dot{\mathcal{H}}^s(\Omega)$ is interpreted as an interpolation space*

$$\dot{\mathcal{H}}^s(\Omega) = \begin{cases} [\dot{\mathcal{H}}^1(\Omega),\ \dot{\mathcal{H}}^2(\Omega)]_{s-1} & 1 < s < 2, \\ [\dot{\mathcal{H}}^0(\Omega),\ \dot{\mathcal{H}}^1(\Omega)]_{s} & 0 < s < 1, \\ [\dot{\mathcal{H}}^{-1}(\Omega),\ \dot{\mathcal{H}}^0(\Omega)]_{s+1} & -1 < s < 0. \end{cases}$$

The main result of this subsection is that $\dot{\mathcal{H}}^s(\Omega)$ is identical with the interpolation space $\mathcal{H}^s(\Omega)$ and the norm (1.3) coincides with the norm (1.1) (with analogous one for $-1 < s < 0$). We begin by comparing these spaces of order $s = -1, 0, 1$. Clearly, every $F \in \mathcal{H}^0(\Omega)$ has a series representation in terms of the orthonormal basis Y_n in $\mathcal{H}^0(\Omega)$, $F = \sum_{n=0}^{\infty}(F, Y_n)_{\Omega} Y_n$ and the norm in $\mathcal{H}^0(\Omega)$ is given by Parseval's identity

$$\|F\|^2_{\mathcal{H}^0} = \sum_{n=0}^{\infty} |(F, Y_n)_{\Omega}|^2 = \|F\|^2_{\dot{\mathcal{H}}^0(\Omega)},$$

which means that $\dot{\mathcal{H}}^0(\Omega) = \mathcal{H}^0(\Omega)$ with the same norm.

The first lemma is concerned with the identification of $\dot{\mathcal{H}}^{-1}(\Omega)$ with the Sobolev space $\mathcal{H}^{-1}(\Omega)$.

Lemma 2.7. *It holds that $\dot{\mathcal{H}}^{-1}(\Omega) = \mathcal{H}^{-1}(\Omega)$. In addition, for $F \in \mathcal{H}^{-1}(\Omega)$,*

$$\|F\|_{\dot{\mathcal{H}}^{-1}(\Omega)} = \|F\|_{\mathcal{H}^{-1}(\Omega)}. \tag{2.5}$$

Proof. For $F \in \dot{\mathcal{H}}^{-1}(\Omega)$ with $F = \sum_{n=0}^{\infty} F_n Y_n$, the relation $Y_n = (1+\lambda_n)^{-1/2}\widetilde{Y}_n$ leads to a series expansion of F

$$F = \sum_{n=0}^{\infty}(1+\lambda_n)^{-1/2} F_n \widetilde{Y}_n \tag{2.6}$$

in terms of the orthonormal basis in $\mathcal{H}^{-1}(\Omega)$. Since $\|F\|^2_{\dot{\mathcal{H}}^{-1}(\Omega)} = \sum_{n=0}^{\infty}(1+\lambda_n)^{-1}|F_n|^2 < \infty$, the series (2.6) converges in $\mathcal{H}^{-1}(\Omega)$ and hence $F \in \mathcal{H}^{-1}(\Omega)$.

Conversely, since $\{\widetilde{Y}_n\}_{n=0}^{\infty}$ is a complete orthonormal basis of $\mathcal{H}^{-1}(\Omega)$, every $F \in \mathcal{H}^{-1}(\Omega)$ has a series representation

$$F = \sum_{n=0}^{\infty} (F, \widetilde{Y}_n)_{-1,\Omega} \widetilde{Y}_n \tag{2.7}$$

converging in $\mathcal{H}^{-1}(\Omega)$ and its norm is evaluated by Parseval's identity,

$$\|F\|^2_{\mathcal{H}^{-1}(\Omega)} = \sum_{n=0}^{\infty} |(F, \widetilde{Y}_n)_{-1,\Omega}|^2. \tag{2.8}$$

Now, we note that by the definition of the $\mathcal{H}^{-1}$-inner product $(\cdot,\cdot)_{-1,\Omega}$

$$\begin{aligned} (F, \widetilde{Y}_n)_{-1,\Omega} &= \langle F, \mathcal{T}(\widetilde{Y}_n)\rangle_{1,\Omega} \\ &= \langle F, (1+\lambda_n)^{-1}\widetilde{Y}_n\rangle_{1,\Omega} = (1+\lambda_n)^{-1/2}\langle F, Y_n\rangle_{1,\Omega}. \end{aligned} \tag{2.9}$$

Noting that $\widetilde{Y}_n = (1+\lambda_n)^{1/2} Y_n$, we substitute (2.9) into (2.7) and (2.8) to show that $F = \sum_{n=0}^{\infty} \langle F, Y_n\rangle_{1,\Omega} Y_n$ and

$$\|F\|^2_{\dot{\mathcal{H}}^{-1}(\Omega)} = \sum_{n=0}^{\infty} (1+\lambda_n)^{-1} |\langle F, Y_n\rangle_{1,\Omega}|^2 = \|F\|^2_{\mathcal{H}^{-1}(\Omega)} < \infty,$$

which implies $F \in \dot{\mathcal{H}}^{-1}(\Omega)$ and (2.5). □

The next lemma is the result for $\mathcal{H}^1(\Omega)$ analogous to the preceding lemma.

Lemma 2.8. *It holds that $\dot{\mathcal{H}}^1(\Omega) = \mathcal{H}^1(\Omega)$. In addition, for $F \in \mathcal{H}^1(\Omega)$,*

$$\|F\|_{\dot{\mathcal{H}}^1(\Omega)} = \|F\|_{\mathcal{H}^1(\Omega)}. \tag{2.10}$$

Proof. For $F \in \dot{\mathcal{H}}^1(\Omega)$ with $F = \sum_{n=0}^{\infty} F_n Y_n$, the relation $Y_n = (1+\lambda_n)^{1/2}\hat{Y}_n$ allows us to have a series expansion of F

$$F = \sum_{n=0}^{\infty} (1+\lambda_n)^{1/2} F_n \hat{Y}_n \tag{2.11}$$

in terms of the orthonormal basis in $\mathcal{H}^1(\Omega)$. Since $\|F\|^2_{\dot{\mathcal{H}}^1(\Omega)} = \sum_{n=0}^{\infty}(1+\lambda_n)|F_n|^2 < \infty$, the series (2.11) converges in $\mathcal{H}^1(\Omega)$ and hence $F \in \mathcal{H}^1(\Omega)$.

Conversely, since $\{\hat{Y}_n\}_{n=0}^{\infty}$ is a complete orthonormal basis of $\mathcal{H}^1(\Omega)$, every $F \in \mathcal{H}^1(\Omega)$ has a series representation

$$F = \sum_{n=0}^{\infty} (F, \hat{Y}_n)_{1,\Omega} \hat{Y}_n \tag{2.12}$$

converging in $\mathcal{H}^1(\Omega)$ and its norm is given by

$$\|F\|^2_{\mathcal{H}^1(\Omega)} = \sum_{n=0}^{\infty} |(F, \hat{Y}_n)_{1,\Omega}|^2. \tag{2.13}$$

A simple computation by using the definition of $\mathcal{T}$ reveals that

$$(F, \hat{Y}_n)_{1,\Omega} = (F, (1+\lambda_n)^{1/2}\mathcal{T}(Y_n))_{1,\Omega} = (1+\lambda_n)^{1/2}(F, Y_n)_{\Omega}. \tag{2.14}$$

Since $Y_n = (1+\lambda_n)^{1/2}\hat{Y}_n$, substituting (2.14) into (2.12) and (2.13) results in the eigenfunction expansion $F = \sum_{n=0}^{\infty}(F, Y_n)_\Omega Y_n$ and

$$\|F\|^2_{\dot{\mathcal{H}}^1(\Omega)} = \sum_{n=0}^{\infty}(1+\lambda_n)|(F, Y_n)_\Omega|^2 = \|F\|^2_{\mathcal{H}^1(\Omega)} < \infty.$$

which shows $F \in \dot{\mathcal{H}}^1(\Omega)$ and (2.10). □

Now, we are ready to establish that $\dot{\mathcal{H}}^s(\Omega) = \mathcal{H}^s(\Omega)$ and two norms coincide for $-1 \le s \le 1$.

Theorem 2.9. *It holds that* $\dot{\mathcal{H}}^s(\Omega) = \mathcal{H}^s(\Omega)$ *for* $-1 \le s \le 1$. *Furthermore, for* $u \in \mathcal{H}^s(\Omega)$,

$$\|u\|_{\dot{\mathcal{H}}^s(\Omega)} = \|u\|_{\mathcal{H}^s(\Omega)}.$$

Proof. Lemma 2.7 shows that $\dot{\mathcal{H}}^{-1}(\Omega) = \mathcal{H}^{-1}(\Omega)$ and two norms coincide. Also Lemma 2.8 gives the same result for $\dot{\mathcal{H}}^1(\Omega)$ and $\mathcal{H}^1(\Omega)$. It is obvious that $\dot{\mathcal{H}}^0(\Omega) = \mathcal{H}^0(\Omega)$ and two norms $\|\cdot\|_{\dot{\mathcal{H}}^0(\Omega)}$ and $\|\cdot\|_{\mathcal{H}^0(\Omega)}$ are identical. Consequently, the result for $-1 < s < 1$ is obtained by the real interpolation technique [15],

$$\dot{\mathcal{H}}^s(\Omega) = [\dot{\mathcal{H}}^{-1}(\Omega),\ \dot{\mathcal{H}}^0(\Omega)]_{1+s} = [\mathcal{H}^{-1}(\Omega),\ \mathcal{H}^0(\Omega)]_{1+s} = \mathcal{H}^s(\Omega)$$

for $-1 < s < 0$, and

$$\dot{\mathcal{H}}^s(\Omega) = [\dot{\mathcal{H}}^0(\Omega),\ \dot{\mathcal{H}}^1(\Omega)]_s = [\mathcal{H}^0(\Omega),\ \mathcal{H}^1(\Omega)]_s = \mathcal{H}^s(\Omega)$$

for $0 < s < 1$. □

2.4. **Fractional Sobolev spaces $\dot{\mathcal{H}}^s(\Omega)$ of order** $1 < s \le 2$**.** Since Y_n has zero Neumann trace on $\partial\Omega$, $\{Y_n\}_{n=0}^{\infty}$ is not dense in $\mathcal{H}^2(\Omega)$. In order to study the spaces spanned by Y_n, we recall the space $\mathcal{H}^2_n(\Omega)$ that is a closed subspace of functions in $\mathcal{H}^2(\Omega)$ with zero Neumann trace on $\partial\Omega$,

$$\mathcal{H}^2_n(\Omega) = \{u \in \mathcal{H}^2(\Omega)\ :\ \partial u/\partial \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega\},$$

and $\mathcal{H}^s_n(\Omega) = [\mathcal{H}^1(\Omega),\ \mathcal{H}^2_n(\Omega)]_{s-1}$ for $1 < s < 2$.

Remark 2.10. *It is shown in* [9, 16] *that* $\mathcal{H}^s_n(\Omega) = \mathcal{H}^s(\Omega)$ *for* $1 \le s < 3/2$, *however,* $\mathcal{H}^s_n(\Omega)$ *for* $3/2 < s \le 2$ *is the subspace of functions in* $\mathcal{H}^s(\Omega)$ *with zero Neumann trace on* $\partial\Omega$. *Since* $\mathcal{H}^s_n(\Omega)$ *for* $3/2 < s \le 2$ *is closed in* $\mathcal{H}^s(\Omega)$, *two norms* $\|\cdot\|_{\mathcal{H}^s_n(\Omega)}$ *and* $\|\cdot\|_{\mathcal{H}^s(\Omega)}$ *are equivalent to each other. For* $s = 3/2$, *the space* $\mathcal{H}^s_n(\Omega)$ *is the set of functions* u *in* $\mathcal{H}^s(\Omega)$ *characterized by the condition* $\rho(x)^{-1/2}|\nabla u| \in L^2(\Omega)$, *where* $\rho(x)$ *is the distance from* x *to the boundary* $\partial\Omega$, *but we remark that* $\mathcal{H}^{3/2}_n(\Omega)$ *is not closed in* $\mathcal{H}^{3/2}(\Omega)$ *(see also* [15]*).*

Lemma 2.11. *It holds that* $\dot{\mathcal{H}}^2(\Omega) = \mathcal{H}^2_n(\Omega)$ *with equivalent norms.*

Proof. Noting that for $u \in \mathcal{H}^2_n(\Omega)$ and $v \in \mathcal{H}^1(\Omega)$,

$$|\langle \mathcal{L}(u), v\rangle_{1,\Omega}| = |(\nabla u, \nabla v)_\Omega + (u, v)_\Omega| = |(-\Delta u + u, v)_\Omega| \le C\|u\|_{\mathcal{H}^2(\Omega)}\|v\|_{\mathcal{H}^0(\Omega)}$$

due to zero Neumann trace of u on $\partial\Omega$, we see that $\mathcal{L} : \mathcal{H}^2_n(\Omega) \to \mathcal{H}^0(\Omega)$ is bounded and

$$\|\mathcal{L}(u)\|_{\mathcal{H}^0(\Omega)} \le C\|u\|_{\mathcal{H}^2(\Omega)}. \tag{2.15}$$

Now, we shall show that $\|\mathcal{L}(u)\|_{\mathcal{H}^0(\Omega)} = \|u\|_{\dot{\mathcal{H}}^2(\Omega)}$, from which together with (2.15) it follows that

$$\|u\|_{\dot{\mathcal{H}}^2(\Omega)} \leq C\|u\|_{\mathcal{H}^2(\Omega)}. \tag{2.16}$$

Let $u = \sum_{n=0}^{\infty} u_n Y_n \in \mathcal{H}^2_n(\Omega)$ (converging at least in $\mathcal{H}^0(\Omega)$). Since $\mathcal{L}(u)$ is in $\mathcal{H}^0(\Omega)$, we can find a series expansion of $\mathcal{L}(u)$ in terms of the orthonormal basis $\{Y_n\}_{n=0}^{\infty}$ in $\mathcal{H}^0(\Omega)$. To do this, we observe that due to zero Neumann trace of Y_n on $\partial\Omega$

$$(\mathcal{L}(u), Y_n)_\Omega = (\nabla u, \nabla Y_n)_\Omega + (u, Y_n)_\Omega = -(u, \Delta Y_n)_\Omega + (u, Y_n)_\Omega = (\lambda_n + 1)u_n,$$

which implies that $\mathcal{L}(u) = \sum_{n=0}^{\infty}(1+\lambda_n)u_n Y_n$. Therefore we can obtain the desired equality

$$\|\mathcal{L}(u)\|_{\mathcal{H}^0(\Omega)} = \|\sum_{n=0}^{\infty}(1+\lambda_n)u_n Y_n\|_{\mathcal{H}^0(\Omega)} = \|u\|_{\dot{\mathcal{H}}^2(\Omega)}.$$

Conversely, for $u \in \dot{\mathcal{H}}^2(\Omega)$ having the series expansion, $u = \sum_{n=0}^{\infty} u_n Y_n$, satisfying $\|u\|^2_{\dot{\mathcal{H}}^2(\Omega)} = \sum_{n=0}^{\infty}(1+\lambda_n)^2|u_n|^2 < \infty$, we first assert that $-\Delta u = \sum_{n=0}^{\infty} \lambda_n u_n Y_n$. Indeed, let $G := \sum_{n=0}^{\infty} \lambda_n u_n Y_n$, which is in $\mathcal{H}^0(\Omega)$. Since the partial sum $U_m = \sum_{n=0}^{m} u_n Y_n$ converges to u in $\mathcal{H}^0(\Omega)$ and $-\Delta U_m$ converges to G in $\mathcal{H}^0(\Omega)$, it can be shown that $-\Delta u = G$. Now the regularity theory for the Neumann boundary value problem, e.g., [10] shows that

$$\|u\|_{\mathcal{H}^2(\Omega)} \leq C\| - \Delta u + u\|_{\mathcal{H}^0(\Omega)} = C\|u\|_{\dot{\mathcal{H}}^2(\Omega)}. \tag{2.17}$$

As a consequence, two inequalities (2.16) and (2.17) establish the equivalence of two norms and show that two spaces $\mathcal{H}^2_n(\Omega)$ and $\dot{\mathcal{H}}^2(\Omega)$ are identical. □

Theorem 2.12. *For $1 \leq s \leq 2$, it holds that $\dot{\mathcal{H}}^s(\Omega) = \mathcal{H}^s_n(\Omega)$ with equivalent norms.*

Proof. By Lemma 2.8 and Lemma 2.11, we have $\dot{\mathcal{H}}^1(\Omega) = \mathcal{H}^1(\Omega)$ and $\dot{\mathcal{H}}^2(\Omega) = \mathcal{H}^2_n(\Omega)$ with equivalent norms. The real interpolation completes to show that $\dot{\mathcal{H}}^s(\Omega) = \mathcal{H}^s_n(\Omega)$ for $1 < s < 2$ and their norms are equivalent. □

2.5. **Application to a cross-sectional trace estimate in waveguides.** Let $\widehat{\Omega}$ be a bounded domain in $\mathbb{R}^d$ for $d = 2$ or 3 such that with a bounded and smooth $\Omega \subset \mathbb{R}^{d-1}$ and a constant $L > 0$,

$$\widehat{\Omega} \cap \{(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1} \ : \ x > -L\} = (-L, 0) \times \Omega$$

and $\partial\widehat{\Omega} \cap \{x < 0\}$ is smooth, which is a typical geometry of semi-infinite waveguides truncated at $x = 0$. We denote $\Gamma_0 := \{0\} \times \Omega$ and $\Gamma_b := \partial\widehat{\Omega} \setminus \bar{\Gamma}_0$ for mutually disjoint parts of the boundary of $\widehat{\Omega}$. In this subsection, we examine a cross-sectional trace estimate of functions whose normal derivative vanishes on Γ_b. The cross-sectional trace estimate in terms of the norm based on eigenfunction expansions is of importance in analyses of acoustic and polarized electromagnetic wave propagations in waveguides [2, 11]. We introduce a trace operator $\gamma(u) = u|_{\Gamma_0}$ for $u \in \mathcal{H}^s(\widehat{\Omega})$ with $1/2 < s \leq 2$.

Theorem 2.13. *If $u \in \mathcal{H}^s(\widehat{\Omega})$ for $1/2 < s \leq 2$ satisfies $\partial u/\partial \boldsymbol{\nu} = 0$ on Γ_b, then $\gamma(u)$ is in $\dot{\mathcal{H}}^{s-1/2}_n(\Gamma_0)$ and satisfies*

$$\|\gamma(u)\|_{\dot{\mathcal{H}}^{s-1/2}(\Gamma_0)} \leq C\|u\|_{\mathcal{H}^s(\widehat{\Omega})}. \tag{2.18}$$

Proof. Since $\mathcal{H}^{s-1/2}(\Gamma_0) = \dot{\mathcal{H}}^{s-1/2}(\Gamma_0)$ for $1/2 < s < 2$ as seen in Theorem 2.9, Remark 2.10 and Theorem 2.12, (2.18) is a standard trace inequality. So we are left with only the case of $s = 2$.

For $s = 2$, we note that $\dot{\mathcal{H}}^{3/2}(\Gamma_0) = \mathcal{H}_n^{3/2}(\Gamma_0)$ is a strict subspace of $\mathcal{H}^{3/2}(\Gamma_0)$ with a finer topology. In this case we prove directly the trace inequality by following the idea as that used in [17]. To do this, let $\Omega_E = \widehat{\Omega} \cup \Gamma_0 \cup \widehat{\Omega}^*$, where $\widehat{\Omega}^*$ is the domain obtained by reflecting $\widehat{\Omega}$ in the y-space and we define an extension operator $E : \mathcal{H}_n^2(\widehat{\Omega}) \to \mathcal{H}_n^2(\Omega_E)$ by $E(u) = \tilde{u}$ for $u \in \mathcal{H}_n^2(\widehat{\Omega})$:

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & \text{for } (x,y) \in \widehat{\Omega}, \\ (a_1 u(-x,y) + a_2 u(-2x,y))\eta(x) & \text{for } (x,y) \in \widehat{\Omega}^* \text{ and } 0 < x < L \\ 0 & \text{for } (x,y) \in \widehat{\Omega}^* \text{ and } x > L \end{cases}$$

with $a_1 + a_2 = 1$ and $-a_1 - 2a_2 = 1$, where η is a smooth cutoff function that is one near Γ_0 and zero for $x > L/2$. Clearly, the extension $\tilde{u}$ is in $\mathcal{H}_n^2(\Omega_E)$ and satisfies $\|\tilde{u}\|_{\mathcal{H}^2(\Omega_E)} \le C\|u\|_{\mathcal{H}^2(\widehat{\Omega})}$.

Now, we will show that for $v \in C_n^\infty(\overline{\Omega_E})$

$$\|\gamma(v)\|_{\dot{\mathcal{H}}^{3/2}(\Gamma_0)} \le C\|v\|_{\mathcal{H}^2(\Omega_E)}, \tag{2.19}$$

where $C_n^\infty(\overline{\Omega_E})$ is a subspace of smooth functions u in $C^\infty(\overline{\Omega_E})$ such that $\partial v/\partial \boldsymbol{\nu} = 0$ on $\partial\Omega_E$. Once we have it, the desired trace inequality immediately follows from the density of $C_n^\infty(\overline{\Omega_E})$ in $\mathcal{H}_n^2(\Omega_E)$ (see Appendix) and the bounded extension operator $E : \mathcal{H}_n^2(\widehat{\Omega}) \to \mathcal{H}_n^2(\Omega_E)$. To prove (2.19) let $v \in C_n^\infty(\overline{\Omega_E})$. By using a cutoff function χ of x which is one for $|x| < L/2$ and vanishes for $|x| > L$, we have the zero extension $\tilde{v}$ of $\chi v|_{(-L,L)\times\Omega}$ to $\Omega_\infty := \mathbb{R} \times \Omega$ such that $\|\tilde{v}\|_{\mathcal{H}^2(\Omega_\infty)} \le C\|v\|_{\mathcal{H}^2(\Omega_E)}$. Also, it has a series representation

$$\tilde{v}(x,y) = \sum_{n=0}^{\infty} \tilde{v}_n(x) Y_n(y) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} \mathcal{F}(\tilde{v}_n)(\xi) e^{-i\xi x} d\xi \right) Y_n(y)$$

for $(x,y) \in \mathbb{R} \times \Omega$, where $\mathcal{F}(\tilde{v}_n)$ is a Fourier transform of $\tilde{v}_n$. Here we note that the derivative of $\tilde{v}$ with respect to x can be interchanged with the infinite sum. Indeed, let $\psi(x) = (\partial\tilde{v}/\partial x(x,\cdot), Y_n)_\Omega$ be the n-th Fourier coefficient of $\partial\tilde{v}/\partial x$. For any $\phi(x) \in C_0^\infty(\mathbb{R})$, by integration by parts

$$\begin{aligned} \int_{\mathbb{R}} \psi(x)\phi(x)dx &= -\int_{\mathbb{R}} \int_{\Omega} \tilde{v}(x,y) \frac{d\phi}{dx}(x) Y_n(y) dy dx \\ &= -\int_{\mathbb{R}} \tilde{v}_n(x) \frac{d\phi}{dx}(x) dx = \int_{\mathbb{R}} \frac{d\tilde{v}_n}{dx}(x)\phi(x)dx, \end{aligned}$$

which shows that $\psi = d\tilde{v}_n/dx$ and hence

$$\frac{\partial\tilde{v}}{\partial x}(x,y) = \sum_{n=0}^{\infty} \frac{d\tilde{v}_n}{dx}(x) Y_n(y).$$

The same argument gives the same result for the second derivative of $\tilde{v}$ with respect to x,

$$\frac{\partial^2\tilde{v}}{\partial x^2}(x,y) = \sum_{n=0}^{\infty} \frac{d^2\tilde{v}_n}{dx^2}(x) Y_n(y).$$

Therefore, by Fubini's theorem and the monotone convergence theorem together with Theorem 2.9 and 2.12 we can show that
(2.20)

$$
\begin{aligned}
\|\tilde{v}\|^2_{\mathcal{H}^2(\Omega_\infty)} &= \int_{\mathbb{R}} \|\tilde{v}(x,\cdot)\|^2_{\mathcal{H}^2(\Omega)} + \|\frac{\partial \tilde{v}}{\partial x}(x,\cdot)\|^2_{\mathcal{H}^1(\Omega)} + \|\frac{\partial^2 \tilde{v}}{\partial x^2}(x,\cdot)\|^2_{\mathcal{H}^0(\Omega)} dx \\
&= \int_{\mathbb{R}} \sum_{n=0}^{\infty} (1+\lambda_n)^2 |\tilde{v}_n(x)|^2 + (1+\lambda_n)|| \frac{d\tilde{v}_n}{dx}(x)|^2 + |\frac{d^2\tilde{v}_n}{dx^2}(x)|^2 dx \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{R}} \big((1+\lambda_n)^2 + (1+\lambda_n)|\xi|^2 + |\xi|^4\big) |\mathcal{F}(\tilde{v}_n)(\xi)|^2 d\xi \\
&\geq \frac{1}{2} \sum_{n=0}^{\infty} \int_{\mathbb{R}} (1+\lambda_n+|\xi|^2)^2 |\mathcal{F}(\tilde{v}_n)(\xi)|^2 d\xi.
\end{aligned}
$$

Now, we shall examine the n-th coefficient of $\gamma(v)$, which is given by

$$
\begin{aligned}
(\gamma(v))_n &= \int_{\mathbb{R}} \mathcal{F}(\tilde{v}_n)(\xi) d\xi \\
&= \int_{\mathbb{R}} (1+\lambda_n+|\xi|^2)^{-1} (1+\lambda_n+|\xi|^2) \mathcal{F}(\tilde{v}_n)(\xi) d\xi.
\end{aligned}
$$

We apply the Cauchy-Schwarz inequality to show that

$$
|(\gamma(v))_n|^2 \leq \int_{\mathbb{R}} (1+\lambda_n+|\xi|^2)^{-2} d\xi \int_{\mathbb{R}} (1+\lambda_n+|\xi|^2)^2 |\mathcal{F}(\tilde{v}_n)(\xi)|^2 d\xi.
$$

Since a change of variables leads to

$$
\begin{aligned}
\int_{\mathbb{R}} (1+\lambda_n+|\xi|^2)^{-2} d\xi &= \frac{1}{(1+\lambda_n)^2} \int_{\mathbb{R}} \left(1 + \left(\frac{\xi}{\sqrt{1+\lambda_n}}\right)^2\right)^{-2} d\xi \\
&= (1+\lambda_n)^{-3/2} \int_{\mathbb{R}} (1+t^2)^{-2} dt,
\end{aligned}
$$

we have that

$$
(1+\lambda_n)^{3/2} |(\gamma(v))_n|^2 \leq C \int_{\mathbb{R}} (1+\lambda_n+|\xi|^2)^2 |\mathcal{F}(\tilde{v}_n)(\xi)|^2 d\xi
$$

and hence by (2.20)

$$
\begin{aligned}
\|\gamma(v)\|^2_{\dot{\mathcal{H}}^{3/2}(\Gamma_0)} &\leq C \sum_{n=0}^{\infty} \int_{\mathbb{R}} (1+\lambda_n+|\xi|^2)^2 |\mathcal{F}(\tilde{v}_n)(\xi)|^2 d\xi \\
&\leq C \|\tilde{v}\|^2_{\mathcal{H}^2(\Omega_\infty)} \leq C \|v\|^2_{\mathcal{H}^2(\Omega_E)},
\end{aligned}
$$

which completes the proof. □

3. Fractional Sobolev spaces of vector-valued functions related to zero tangential component or zero normal component

In this section, for a simply-connected, bounded and smooth domain $\Omega \subset \mathbb{R}^2$, we study fractional Sobolev spaces of vector-valued functions related to the boundary condition of zero tangential component or zero normal component in Ω. To do this, denoting $\boldsymbol{H}^1(\Omega) = (H^1(\Omega))^2$, we define

$$\boldsymbol{H}^1_T(\Omega) := \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \ : \ \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \quad \text{on } \partial\Omega\},$$
$$\boldsymbol{H}^1_N(\Omega) := \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \ : \ \boldsymbol{\nu}^\perp \cdot \boldsymbol{u} = 0 \quad \text{on } \partial\Omega\},$$

where $\boldsymbol{\nu}^\perp = R\boldsymbol{\nu}$ with R rotation by $-90°$. Hereafter we will use boldface to represent vector-valued functions or operators/spaces of vector-valued functions. Let $\boldsymbol{H}^{-1}_T(\Omega)$ and $\boldsymbol{H}^{-1}_N(\Omega)$ be the dual spaces of $\boldsymbol{H}^1_T(\Omega)$ and $\boldsymbol{H}^1_N(\Omega)$ with the pivot space $\boldsymbol{L}^2(\Omega)$, respectively. Their duality pairings are denoted by $\langle \cdot, \cdot \rangle_{1,T,\Omega}$ and $\langle \cdot, \cdot \rangle_{1,N,\Omega}$. The operator norms in $\boldsymbol{H}^1_*(\Omega)$ are given by

$$\|\boldsymbol{F}\|_{\boldsymbol{H}^{-1}_*(\Omega)} := \sup_{0 \neq \boldsymbol{v} \in \boldsymbol{H}^1_*(\Omega)} \frac{|\langle \boldsymbol{F}, \boldsymbol{v} \rangle_{1,*,\Omega}|}{\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}}, \tag{3.1}$$

where $*$ stands for T or N. We define intermediate spaces by the interpolation

$$\boldsymbol{H}^s_*(\Omega) := \begin{cases} [\boldsymbol{L}^2(\Omega),\ \boldsymbol{H}^1_*(\Omega)]_s & 0 \le s \le 1, \\ [\boldsymbol{H}^{-1}_*(\Omega),\ \boldsymbol{L}^2(\Omega)]_{1+s} & -1 \le s \le 0, \end{cases}$$

with $* = T$ or $* = N$. The duality pairing between $\boldsymbol{H}^{-s}_*(\Omega)$ and $\boldsymbol{H}^s_*(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle_{s,*,\Omega}$. We will characterize the interpolation spaces in terms of a norm based on series expansions of eigenfunctions with zero normal component or zero tangential component of the vector Laplacian.

3.1. **Eigenvalue problems of the vector Laplacian.** Let us consider the eigenvalue problems for the vector Laplacian defined in $\Omega \subset \mathbb{R}^2$,

$$\begin{aligned} -\boldsymbol{\nabla}\nabla \cdot \boldsymbol{u} + \boldsymbol{\nabla}^\perp \nabla^\perp \cdot \boldsymbol{u} &= \eta \boldsymbol{u} \quad \text{in } \Omega, \\ \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \ \text{ and } \ \nabla^\perp \cdot \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} -\boldsymbol{\nabla}\nabla \cdot \boldsymbol{u} + \boldsymbol{\nabla}^\perp \nabla^\perp \cdot \boldsymbol{u} &= \eta \boldsymbol{u} \quad \text{in } \Omega, \\ \boldsymbol{\nu}^\perp \cdot \boldsymbol{u} = 0 \ \text{ and } \ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.3}$$

where $\nabla^\perp \cdot$ and $\boldsymbol{\nabla}^\perp$ are scalar- and vector-curl operators defined by

$$\nabla^\perp \cdot \boldsymbol{u} = \nabla \cdot R\boldsymbol{u} \quad \text{and} \quad \boldsymbol{\nabla}^\perp u = R\boldsymbol{\nabla} u,$$

respectively. The first eigenvalue problem (3.2) seeks for eigenfunctions with zero normal component. In contrast, the second eigenvalue problem (3.3) treats eigenfunctions with zero tangential component. These two types of eigenfunctions play an essential role in studying a series representation of time-harmonic electromagnetic fields in perfectly conducting waveguides. In particular, eigenfunctions with zero normal/tangential component are useful to represent cross-sectional traces of electric/magnetic fields in a series form, respectively [12].

We note that weak solutions to the problem (3.2) and (3.3) belong to the spaces $\boldsymbol{H}(curl, \Omega) \cap \boldsymbol{H}_0(div, \Omega)$ and $\boldsymbol{H}_0(curl, \Omega) \cap \boldsymbol{H}(div, \Omega)$, respectively. Since Ω is assumed to be smooth, a regularity theory, e.g., in [8] shows that

$$\boldsymbol{H}^1_T(\Omega) = \boldsymbol{H}(curl, \Omega) \cap \boldsymbol{H}_0(div, \Omega) \quad \text{and} \quad \boldsymbol{H}^1_N(\Omega) = \boldsymbol{H}_0(curl, \Omega) \cap \boldsymbol{H}(div, \Omega)$$

with equivalent norms in the space identities. Thus the spaces $\boldsymbol{H}^1_T(\Omega)$ and $\boldsymbol{H}^1_N(\Omega)$ can be equipped with norm

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}^1(\Omega)} = \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla^\perp \cdot \boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla \cdot \boldsymbol{u}\|^2_{L^2(\Omega)}$$

for $\boldsymbol{u} \in \boldsymbol{H}_T^1(\Omega)$ or $\boldsymbol{H}_N^1(\Omega)$.

3.2. **The space $\boldsymbol{H}_T^s(\Omega)$ for $-1 \le s \le 1$.** The eigenvalue problem (3.2) is reformulated to a weak form in the solution space $\boldsymbol{H}_T^1(\Omega)$: finding $\eta \in \mathbb{R}$ and $\boldsymbol{u} \in \boldsymbol{H}_T^1(\Omega)$ such that

$$A(\boldsymbol{u}, \boldsymbol{v}) := (\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_\Omega + (\nabla^\perp \cdot \boldsymbol{u}, \nabla^\perp \cdot \boldsymbol{v})_\Omega = \eta(\boldsymbol{u}, \boldsymbol{v})_\Omega \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_T^1(\Omega). \tag{3.4}$$

By the Helmholtz decomposition [8], eigenfunctions $\boldsymbol{u}$ and test functions $\boldsymbol{v}$ in $\boldsymbol{H}_T^1(\Omega)$ can be decomposed into

$$\boldsymbol{u} = \boldsymbol{\nabla} Y + \boldsymbol{\nabla}^\perp V \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{\nabla}\Psi + \boldsymbol{\nabla}^\perp \Phi,$$

where $V, \Phi \in H_0^1(\Omega)$ and $Y, \Psi \in H^1(\Omega)$ with $\partial Y/\partial \boldsymbol{\nu}, \partial\Psi/\partial\boldsymbol{\nu} = 0$ on $\partial\Omega$.

Now, noting that the set of Dirichlet eigenfunctions V_n for eigenvalues μ_n, $n = 1, 2, \dots$ is dense in $H_0^1(\Omega)$ and $L^2(\Omega)$ as shown in [3], by taking test functions $\boldsymbol{v} = \boldsymbol{\nabla}^\perp V_n$, we see that

$$(-\Delta V, V_n)_\Omega = \eta(V, V_n)_\Omega$$

from (3.4), which implies that V is a Dirichlet eigenfunction and η is a Dirichlet eigenvalue. In such a case, since $A(\boldsymbol{\nabla}^\perp V, \boldsymbol{\nabla}\Psi) = 0$ and $(\boldsymbol{\nabla}^\perp V, \boldsymbol{\nabla}\Psi)_\Omega = 0$ for all $\Psi \in H^1(\Omega)$, we have $A(\boldsymbol{\nabla}^\perp V, \boldsymbol{v}) = \eta(\boldsymbol{\nabla}^\perp V, \boldsymbol{v})_\Omega$ for all $\boldsymbol{v} \in \boldsymbol{H}_T^1(\Omega)$, which shows that $\boldsymbol{\nabla}^\perp V_n$ and μ_n for $n \ge 1$ are eigenpairs of the eigenvalue problem (3.4).

On the other hand, at this time we use the fact that the set of Neumann eigenfunctions Y_n for eigenvalues λ_n, $n = 0, 1, \dots$ is dense in $H^1(\Omega)$ and $L^2(\Omega)$ shown by Lemma 2.5 and Lemma 2.4 and take test functions $\boldsymbol{v} = \boldsymbol{\nabla} Y_n$ to obtain that

$$(-\Delta Y, Y_n)_\Omega = \eta(Y, Y_n)_\Omega,$$

from which it then follows that Y is a Neumann eigenfunction and η is a Neumann eigenvalue except for $\eta = 0$. Since $A(\boldsymbol{\nabla} Y, \boldsymbol{\nabla}^\perp\Phi) = 0$ and $(\boldsymbol{\nabla} Y, \boldsymbol{\nabla}^\perp \Phi)_\Omega = 0$ for $\Phi \in H_0^1(\Omega)$, it can be shown that $\boldsymbol{\nabla}^\perp Y_n$ and λ_n for $n \ge 1$ are eigenpairs of the eigenvalue problem (3.4). As a conclusion, we have the following proposition.

Proposition 3.1. *The complete set of eigenvalues of the problem* (3.3) *is given by* $\{\lambda_n\}_{n=1}^\infty \cup \{\mu_n\}_{n=1}^\infty$, *the set of non-zero Neumann eigenvalues and Dirichlet eigenvalues of the Laplacian, and their corresponding eigenfunctions are* $\{\boldsymbol{\nabla} Y_n\}_{n=1}^\infty$ *and* $\{\boldsymbol{\nabla}^\perp V_n\}_{n=1}^\infty$.

We can develop the same theory as done for the Neumann Laplacian in the preceding section. We start by defining $\mathbb{L}_T : \boldsymbol{H}_T^1(\Omega) \to \boldsymbol{H}_T^{-1}(\Omega)$ pertaining to the weak vector Laplacian by

$$\langle \mathbb{L}_T(\boldsymbol{u}), \boldsymbol{v}\rangle_{1,T,\Omega} = (\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_\Omega + (\nabla^\perp \cdot \boldsymbol{u}, \nabla^\perp \cdot \boldsymbol{v})_\Omega + (\boldsymbol{u}, \boldsymbol{v})_\Omega := (\boldsymbol{u}, \boldsymbol{v})_{1,\Omega}$$

for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_T^1(\Omega)$. For the sake of simplicity, we abuse the notation $(\cdot,\cdot)_{1,\Omega}$ for the $\boldsymbol{H}^1(\Omega)$-inner product of vector-valued functions but it can be clearly distinguished from the $H^1(\Omega)$-inner product of scalar-valued functions from context. Due to the Lax-Milgram lemma, its inverse operator $\mathbb{T}_T : \boldsymbol{H}_T^{-1}(\Omega) \to \boldsymbol{H}_T^1(\Omega)$ is also well-defined by

$$(\mathbb{T}_T(\boldsymbol{F}), \boldsymbol{v})_{1,\Omega} = \langle \boldsymbol{F}, \boldsymbol{v}\rangle_{1,T,\Omega} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_T^1(\Omega)$$

for $\boldsymbol{F} \in \boldsymbol{H}_T^{-1}(\Omega)$. By using the inverse operator $\mathbb{T}_T$, the inner product $(\cdot,\cdot)_{-1,T,\Omega}$ in $\boldsymbol{H}_T^{-1}(\Omega)$ can be defined as

$$(\boldsymbol{F}, \boldsymbol{G})_{-1,T,\Omega} := \langle \boldsymbol{F}, \mathbb{T}_T(\boldsymbol{G})\rangle_{1,T,\Omega} \text{ for } \boldsymbol{F}, \boldsymbol{G} \in \boldsymbol{H}_T^{-1}(\Omega),$$

which induces the same norm as the operator norm (3.1) and makes $\mathbb{T}_T : \boldsymbol{H}_T^{-1}(\Omega) \to \boldsymbol{H}_T^1(\Omega)$ an isometry,

$$(\boldsymbol{F}, \boldsymbol{G})_{-1,T,\Omega} = (\mathbb{T}_T(\boldsymbol{F}), \mathbb{T}_T(\boldsymbol{G}))_{1,\Omega} \quad \text{for all } \boldsymbol{F}, \boldsymbol{G} \in \boldsymbol{H}_T^{-1}(\Omega).$$

Then $\mathbb{T}_T : \boldsymbol{H}_T^{-1}(\Omega) \to \boldsymbol{H}_T^1(\Omega) \subset \boldsymbol{H}_T^{-1}(\Omega)$ and $\mathbb{T}_T : \boldsymbol{L}^2(\Omega) \to \boldsymbol{H}_T^1(\Omega) \subset \boldsymbol{L}^2(\Omega)$ are continuous, compact and self-adjoint operators in $\boldsymbol{H}_T^{-1}(\Omega)$ and $\boldsymbol{L}^2(\Omega)$, respectively. In addition, $\{(1+\lambda_n)^{-1}, (1+\mu_n)^{-1}\}_{n=1}^\infty$ is the complete set of eigenvalues of $\mathbb{T}_T$ for eigenfunctions $\boldsymbol{\nabla} Y_n$ and $\boldsymbol{\nabla}^\perp V_n$. It brings the following result.

Lemma 3.2. *Let Y_n be orthonormal Neumann eigenfunctions in $L^2(\Omega)$ of the Laplacian for eigenvalues λ_n and V_n be orthonormal Dirichlet eigenfunctions in $L^2(\Omega)$ of the Laplacian for eigenvalues μ_n. We also denote $\boldsymbol{Y}_n = \lambda_n^{-1/2} \boldsymbol{\nabla} Y_n$ and $\boldsymbol{V}_n^\perp = \mu_n^{-1/2} \boldsymbol{\nabla}^\perp V_n$ for $n = 1, 2, \ldots$.*

1. *The set $\{\boldsymbol{Y}_n, \boldsymbol{V}_n^\perp\}_{n=1}^\infty$ is a complete orthonormal basis consisting of eigenfunctions to the problem (3.2) for $\boldsymbol{L}^2(\Omega)$.*
2. *The set $\{(1+\lambda_n)^{-1/2}\boldsymbol{Y}_n, (1+\mu_n)^{-1/2}\boldsymbol{V}_n^\perp\}_{n=1}^\infty$ is a complete orthonormal basis consisting of eigenfunctions to the problem (3.2) for $\boldsymbol{H}_T^1(\Omega)$.*
3. *The set $\{(1+\lambda_n)^{1/2}\boldsymbol{Y}_n, (1+\mu_n)^{1/2}\boldsymbol{V}_n^\perp\}_{n=1}^\infty$ is a complete orthonormal basis consisting of eigenfunctions to the problem (3.2) for $\boldsymbol{H}_T^{-1}(\Omega)$.*

Proof. Every assertion in this lemma except for normalization of eigenfunctions can be proved by the same way as in the previous section based on the spectral theory. Normalization is also easily verified by computing norms of eigenfunctions,

$$\begin{aligned}
\|\boldsymbol{\nabla} Y_n\|^2_{\boldsymbol{L}^2(\Omega)} &= (\boldsymbol{\nabla} Y_n, \boldsymbol{\nabla} Y_n)_\Omega = (-\Delta Y_n, Y_n)_\Omega = \lambda_n, \\
\|\boldsymbol{\nabla} Y_n\|^2_{\boldsymbol{H}^1(\Omega)} &= (\Delta Y_n, \Delta Y_n)_\Omega + (\boldsymbol{\nabla} Y_n, \boldsymbol{\nabla} Y_n)_\Omega = \lambda_n(1+\lambda_n), \\
\|\boldsymbol{\nabla} Y_n\|^2_{\boldsymbol{H}_T^{-1}(\Omega)} &= \|\mathbb{T}_T(\boldsymbol{\nabla} Y_n)\|^2_{\boldsymbol{H}^1(\Omega)} = \lambda_n(1+\lambda_n)^{-1}
\end{aligned}$$

and the same calculations for V_n with λ_n replaced by μ_n. □

Theorem 3.3. *The interpolation space $\boldsymbol{H}_T^s(\Omega)$, $-1 \le s \le 1$, is the space of functions $\boldsymbol{F} = \sum_{n=1}^\infty A_n \boldsymbol{Y}_n + B_n \boldsymbol{V}_n^\perp$ satisfying*

$$\|\boldsymbol{F}\|^2_{\boldsymbol{H}_T^s(\Omega)} := \sum_{n=1}^\infty (1+\lambda_n)^s |A_n|^2 + (1+\mu_n)^s |B_n|^2 < \infty.$$

Proof. The case for $s = 0$ is obvious since $\boldsymbol{Y}_n$ and $\boldsymbol{V}_n^\perp$ form an orthonormal basis of $\boldsymbol{L}^2(\Omega)$. For $s = -1$, let $\widetilde{\boldsymbol{Y}}_n = (1+\lambda_n)^{1/2} \boldsymbol{Y}_n$ and $\widetilde{\boldsymbol{V}}_n^\perp = (1+\mu_n)^{1/2} \boldsymbol{V}_n^\perp$. Since $\{\widetilde{\boldsymbol{Y}}_n, \widetilde{\boldsymbol{V}}_n^\perp\}_{n=1}^\infty$ is an orthonormal basis of $\boldsymbol{H}_T^{-1}(\Omega)$ by Lemma 3.2, any $\boldsymbol{F} \in \boldsymbol{H}_T^{-1}(\Omega)$ can be written as

$$\boldsymbol{F} = \sum_{n=1}^\infty \widetilde{A}_n \widetilde{\boldsymbol{Y}}_n + \widetilde{B}_n \widetilde{\boldsymbol{V}}_n^\perp.$$

Denoting $A_n = (\lambda_n + 1)^{1/2} \widetilde{A}_n$ and $B_n = (\mu_n + 1)^{1/2} \widetilde{B}_n$, it can be shown that

$$\boldsymbol{F} = \sum_{n=1}^\infty A_n \boldsymbol{Y}_n + B_n \boldsymbol{V}_n^\perp$$

and

$$\|\boldsymbol{F}\|^2_{\boldsymbol{H}_T^{-1}(\Omega)} = \sum_{n=0}^{\infty} |\widetilde{A}_n|^2 + |\widetilde{B}_n|^2 = \sum_{n=0}^{\infty} (\lambda_n+1)^{-1}|A_n|^2 + (\mu_n+1)^{-1}|B_n|^2 < \infty.$$

Similarly, by using the orthonormal basis $\{(1+\lambda_n)^{1/2}\,\boldsymbol{Y}_n, (1+\mu_n)^{1/2}\,\boldsymbol{V}_n^{\perp}\}_{n=1}^{\infty}$ of $\boldsymbol{H}_T^1(\Omega)$ we can derive the result for $s = 1$, that is $\boldsymbol{H}_T^1(\Omega)$ is the space of $\boldsymbol{F} = \sum_{n=1}^{\infty} A_n\,\boldsymbol{Y}_n + B_n\,\boldsymbol{V}_n^{\perp}$ satisfying

$$\|\boldsymbol{F}\|^2_{\boldsymbol{H}^1(\Omega)} = \sum_{n=1}^{\infty} (1+\lambda_n)|A_n|^2 + (1+\mu_n)|B_n|^2 < \infty.$$

Finally, the other cases for $-1 < s < 0$ and $0 < s < 1$ follow from the real interpolation. □

3.3. **The space $\boldsymbol{H}_N^s(\Omega)$ for $-1 \le s \le 1$.** The eigenvalue problem (3.3) is reformulated to a weak form in the solution space $\boldsymbol{H}_N^1(\Omega)$: finding $\eta \in \mathbb{R}$ and $\boldsymbol{u} \in \boldsymbol{H}_N^1(\Omega)$ such that

$$(\nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v})_\Omega + (\nabla^\perp \cdot \boldsymbol{u}, \nabla^\perp \cdot \boldsymbol{v})_\Omega = \eta(\boldsymbol{u}, \boldsymbol{v})_\Omega \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_N^1(\Omega). \tag{3.5}$$

In this case, we use the Helmholtz decomposition in [4] showing that eigenfunctions $\boldsymbol{u}$ and test functions $\boldsymbol{v}$ in $\boldsymbol{H}_N^1(\Omega)$ can be decomposed into

$$\boldsymbol{u} = \boldsymbol{\nabla} V + \boldsymbol{\nabla}^\perp Y \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}^\perp \Psi,$$

where $V, \Phi \in H_0^1(\Omega)$ and $Y, \Psi \in H^1(\Omega)$ with $\partial Y/\partial\boldsymbol{\nu}, \partial\Psi/\partial\boldsymbol{\nu} = 0$ on $\partial\Omega$. The same arguments as those used for the case of $\boldsymbol{H}_T^s(\Omega)$ can be carried over without any essential change. We summarize the results for $\boldsymbol{H}_N^s$.

Proposition 3.4. *The complete set of eigenvalues of the problem* (3.3) *is given by* $\{\lambda_n\}_{n=1}^{\infty} \cup \{\mu_n\}_{n=1}^{\infty}$, *the set of non-zero Neumann eigenvalues and Dirichlet eigenvalues of the Laplacian, and their corresponding eigenfunctions are* $\{\boldsymbol{\nabla}^\perp Y_n\}_{n=1}^{\infty}$ *and* $\{\boldsymbol{\nabla} V_n\}_{n=1}^{\infty}$.

Lemma 3.5. *Let Y_n be orthonormal Neumann eigenfunctions in $L^2(\Omega)$ of the Laplacian for eigenvalues λ_n and V_n be orthonormal Dirichlet eigenfunctions in $L^2(\Omega)$ of the Laplacian for eigenvalues μ_n. We also denote $\boldsymbol{Y}_n^\perp = \lambda_n^{-1/2}\boldsymbol{\nabla}^\perp Y_n$ and $\boldsymbol{V}_n = \mu_n^{-1/2}\boldsymbol{\nabla} V_n$ for $n = 1, 2, \ldots$.*

1. *The set $\{\boldsymbol{Y}_n^\perp, \boldsymbol{V}_n\}_{n=1}^{\infty}$ is a complete orthonormal basis consisting of eigenfunctions to the problem* (3.3) *for $\boldsymbol{L}^2(\Omega)$.*
2. *The set $\{(1+\lambda_n)^{-1/2}\boldsymbol{Y}_n^\perp, (1+\mu_n)^{-1/2}\boldsymbol{V}_n\}_{n=1}^{\infty}$ is a complete orthonormal basis consisting of eigenfunctions to the problem* (3.3) *for $\boldsymbol{H}_N^1(\Omega)$.*
3. *The set $\{(1+\lambda_n)^{1/2}\boldsymbol{Y}_n^\perp, (1+\mu_n)^{1/2}\boldsymbol{V}_n\}_{n=1}^{\infty}$ is a complete orthonormal basis consisting of eigenfunctions to the problem* (3.3) *for $\boldsymbol{H}_N^{-1}(\Omega)$.*

Theorem 3.6. *The interpolation space $\boldsymbol{H}_N^s(\Omega)$, $-1 \le s \le 1$, is the space of functions $\boldsymbol{F} = \sum_{n=1}^{\infty} A_n\,\boldsymbol{Y}_n^\perp + B_n\,\boldsymbol{V}_n$ satisfying*

$$\|\boldsymbol{F}\|^2_{\boldsymbol{H}_N^s(\Omega)} := \sum_{n=1}^{\infty} (1+\lambda_n)^s|A_n|^2 + (1+\mu_n)^s|B_n|^2 < \infty.$$

3.4. **Application to cross-sectional trace estimates in waveguides.** For the domain $\widehat{\Omega}$ introduced in Subsection 2.5 with $d = 3$, we define

$$\boldsymbol{H}_b(curl, \widehat{\Omega}) = \{\boldsymbol{u} \in \boldsymbol{H}(curl, \widehat{\Omega}) \ : \ \boldsymbol{\nu} \times \boldsymbol{u} = 0 \text{ on } \Gamma_b\}.$$

In this subsection, we will use the convenient norms based on eigenfunction expansions to analyze the cross-sectional tangential trace and tangential component trace of vector-fields $\boldsymbol{u} \in \boldsymbol{H}_b(curl, \widehat{\Omega})$ on Γ_0 (here Γ_0 is identified with Ω). These cross-sectional tangential trace and tangential component trace are of importance in studying electromagnetic wave propagation in perfectly conducting waveguides [12]. To this end, we first study the regularity estimates for the divergence and curl operators on the cross-sectional boundary Γ_0. Let

$$\mathrm{div}_{\Gamma_0} : \boldsymbol{L}^2(\Gamma_0) \to \dot{\mathcal{H}}^{-1}(\Gamma_0) \ \text{ and } \ \mathrm{curl}_{\Gamma_0} : \boldsymbol{L}^2(\Gamma_0) \to \dot{\mathcal{H}}^{-1}(\Gamma_0)$$

be the surface divergence and curl operators in a weak sense defined by

$$\langle \mathrm{div}_{\Gamma_0} \boldsymbol{\phi}, \psi \rangle_{-1,\Gamma_0} = -(\boldsymbol{\phi}, \boldsymbol{\nabla}_y \psi)_{\Gamma_0},$$
$$\langle \mathrm{curl}_{\Gamma_0} \boldsymbol{\phi}, \psi \rangle_{-1,\Gamma_0} = (\boldsymbol{\phi}, \boldsymbol{\nabla}_y^{\perp} \psi)_{\Gamma_0}$$

for $\boldsymbol{\phi} \in \boldsymbol{L}^2(\Gamma_0)$ and $\psi \in \dot{\mathcal{H}}^1(\Gamma_0)$. Here the subscript y of the operators $\boldsymbol{\nabla}_y$ and $\boldsymbol{\nabla}_y^{\perp}$ is used to indicate that they are differential operators of the variable y on the surface $\Gamma_0 \subset \mathbb{R}^2$. Clearly, it holds that

$$\|\mathrm{div}_{\Gamma_0} \boldsymbol{\phi}\|_{\dot{\mathcal{H}}^{-1}(\Gamma_0)} \le \|\boldsymbol{\phi}\|_{\boldsymbol{L}^2(\Gamma_0)} \ \text{ and } \ \|\mathrm{curl}_{\Gamma_0} \boldsymbol{\phi}\|_{\dot{\mathcal{H}}^{-1}(\Gamma_0)} \le \|\boldsymbol{\phi}\|_{\boldsymbol{L}^2(\Gamma_0)}. \tag{3.6}$$

Now, the surface divergence and curl operators have the following regularity properties.

Lemma 3.7. *For $0 \le s \le 1$, $\mathrm{div}_{\Gamma_0} \boldsymbol{\phi}$ for $\boldsymbol{\phi} \in \boldsymbol{H}_T^s(\Gamma_0)$ is in $\dot{\mathcal{H}}^{s-1}(\Gamma_0)$ and satisfies*

$$\|\mathrm{div}_{\Gamma_0} \boldsymbol{\phi}\|_{\dot{\mathcal{H}}^{s-1}(\Gamma_0)} \le \|\boldsymbol{\phi}\|_{\boldsymbol{H}_T^s(\Gamma_0)}. \tag{3.7}$$

For $-1 \le s \le 0$, there exists a continuous extension $\mathrm{div}_{\Gamma_0} : \boldsymbol{H}_T^s(\Gamma_0) \to \dot{\mathcal{H}}^{s-1}(\Gamma_0)$ satisfying

$$\|\mathrm{div}_{\Gamma_0} \boldsymbol{\phi}\|_{\dot{\mathcal{H}}^{s-1}(\Gamma_0)} \le \|\boldsymbol{\phi}\|_{\boldsymbol{H}_T^s(\Gamma_0)}, \tag{3.8}$$

where $\dot{\mathcal{H}}^{s-1}(\Gamma_0)$ is the dual space of $\dot{\mathcal{H}}^{1-s}(\Gamma_0)$.

Proof. Let $0 \le s \le 1$. For $\boldsymbol{\phi} \in \boldsymbol{H}_T^1(\Gamma_0)$ and $\psi \in \dot{\mathcal{H}}^1(\Gamma_0)$, by the definition of div_{Γ_0} and integration by parts with the boundary condition $\boldsymbol{\nu} \cdot \boldsymbol{\phi} = 0$ on $\partial\Gamma_0$ we observe that

$$\langle \mathrm{div}_{\Gamma_0} \boldsymbol{\phi}, \psi \rangle_{1,\Gamma_0} = -(\boldsymbol{\phi}, \boldsymbol{\nabla}_y \psi)_{\Gamma_0} = (\nabla_y \cdot \boldsymbol{\phi}, \psi)_{\Gamma_0} = \langle \nabla_y \cdot \boldsymbol{\phi}, \psi \rangle_{1,\Gamma_0},$$

which shows that $\mathrm{div}_{\Gamma_0} \boldsymbol{\phi} = \nabla_y \cdot \boldsymbol{\phi}$ is in $\dot{\mathcal{H}}^0(\Gamma_0)$ and $\mathrm{div}_{\Gamma_0} : \boldsymbol{H}_T^1(\Gamma_0) \to \dot{\mathcal{H}}^0(\Gamma_0)$ is a continuous operator satisfying

$$\|\mathrm{div}_{\Gamma_0} \boldsymbol{\phi}\|_{\dot{\mathcal{H}}^0(\Gamma_0)} \le \|\boldsymbol{\phi}\|_{\boldsymbol{H}^1(\Gamma_0)}. \tag{3.9}$$

The real interpolation theory applied to (3.6) and (3.9) establishes (3.7).

For $-1 \le s \le 0$, the operator $\mathrm{div}_{\Gamma_0} : \boldsymbol{L}^2(\Gamma_0) \to \dot{\mathcal{H}}^{-1}(\Gamma_0)$ can be extended to $\boldsymbol{H}_T^{-1}(\Gamma_0)$ by defining $\mathrm{div}_{\Gamma_0} : \boldsymbol{H}_T^{-1}(\Gamma_0) \to \dot{\mathcal{H}}^{-2}(\Gamma_0)$ by

$$\langle \mathrm{div}_{\Gamma_0} \boldsymbol{\phi}, \psi \rangle_{2,\Gamma_0} = -\langle \boldsymbol{\phi}, \boldsymbol{\nabla}_y \psi \rangle_{1,T,\Gamma_0}$$

for $\psi \in \dot{\mathcal{H}}^2(\Gamma_0)$, where $\dot{\mathcal{H}}^{-2}(\Gamma_0)$ is the dual space of $\dot{\mathcal{H}}^2(\Gamma_0) = \mathcal{H}^2_n(\Gamma_0)$ and $\langle \cdot, \cdot \rangle_{2,\Gamma_0}$ is the duality pairing between $\dot{\mathcal{H}}^{-2}(\Gamma_0)$ and $\dot{\mathcal{H}}^2(\Gamma_0)$. For $\psi \in \dot{\mathcal{H}}^2(\Gamma_0)$ with $\psi = \sum_{n=0}^{\infty} \psi_n Y_n$, since $\sum_{n=1}^{\infty} \sqrt{\lambda_n}\psi_n \boldsymbol{Y}_n$ converges in $\boldsymbol{L}^2(\Gamma_0)$ it can be easily shown that

$$\boldsymbol{\nabla}_y \psi = \sum_{n=1}^{\infty} \psi_n \boldsymbol{\nabla}_y Y_n = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\psi_n \boldsymbol{Y}_n$$

and so $\|\boldsymbol{\nabla}_y \psi\|^2_{\boldsymbol{H}^1(\Gamma_0)} = \sum_{n=1}^{\infty} (1+\lambda_n)\lambda_n |\psi_n| < \|\psi\|^2_{\dot{\mathcal{H}}^2(\Gamma_0)}$. Thus, it is well-defined since $\boldsymbol{\nabla}_y \psi$ belongs to $\boldsymbol{H}^1_T(\Gamma_0)$ for $\psi \in \dot{\mathcal{H}}^2(\Gamma_0)$. Also, it is an extension since

$$\langle \mathrm{div}_{\Gamma_0} \boldsymbol{\phi}, \psi \rangle_{2,\Gamma_0} = -\langle \boldsymbol{\phi}, \boldsymbol{\nabla}_y \psi \rangle_{1,T,\Gamma_0} = -(\boldsymbol{\phi}, \boldsymbol{\nabla}_y \psi)_{\Gamma_0}$$

for $\boldsymbol{\phi} \in \boldsymbol{L}^2(\Gamma_0)$.

By estimating the duality pairing

$$|\langle \mathrm{div}_{\Gamma_0} \boldsymbol{\phi}, \psi \rangle_{2,\Gamma_0}| \leq \|\boldsymbol{\phi}\|_{\boldsymbol{H}^{-1}_T(\Gamma_0)} \|\boldsymbol{\nabla}_y \psi\|_{\boldsymbol{H}^1(\Gamma_0)} \leq \|\boldsymbol{\phi}\|_{\boldsymbol{H}^{-1}_T(\Gamma_0)} \|\psi\|_{\dot{\mathcal{H}}^2(\Gamma_0)},$$

we can obtain that

$$\|\mathrm{div}_{\Gamma_0} \boldsymbol{\phi}\|_{\dot{\mathcal{H}}^{-2}(\Gamma_0)} = \sup_{0 \neq \psi \in \dot{\mathcal{H}}^2(\Gamma_0)} \frac{|\langle \mathrm{div}_{\Gamma_0} \boldsymbol{\phi}, \psi \rangle_{2,\Gamma_0}|}{\|\psi\|_{\dot{\mathcal{H}}^2(\Gamma_0)}} \leq \|\boldsymbol{\phi}\|_{\boldsymbol{H}^{-1}_T(\Gamma_0)}. \tag{3.10}$$

Finally, we use the real interpolation theory again to obtain (3.8) from (3.6) and (3.10). □

The regularity estimate of the operator curl_{Γ_0} can be obtained by using the same argument as that used in the above lemma.

Lemma 3.8. *For* $0 \leq s \leq 1$, $\mathrm{curl}_{\Gamma_0} \boldsymbol{\phi}$ *for* $\boldsymbol{\phi} \in \boldsymbol{H}^s_N(\Gamma_0)$ *is in* $\dot{\mathcal{H}}^{s-1}(\Gamma_0)$ *and satisfies*

$$\|\mathrm{curl}_{\Gamma_0} \boldsymbol{\phi}\|_{\dot{\mathcal{H}}^{s-1}(\Gamma_0)} \leq \|\boldsymbol{\phi}\|_{\boldsymbol{H}^s_N(\Gamma_0)}. \tag{3.11}$$

For $-1 \leq s \leq 0$, *there exists a continuous extension* $\mathrm{curl}_{\Gamma_0} : \boldsymbol{H}^s_N(\Gamma_0) \to \dot{\mathcal{H}}^{s-1}(\Gamma_0)$ *satisfying*

$$\|\mathrm{curl}_{\Gamma_0} \boldsymbol{\phi}\|_{\dot{\mathcal{H}}^{s-1}(\Gamma_0)} \leq \|\boldsymbol{\phi}\|_{\boldsymbol{H}^s_N(\Gamma_0)}, \tag{3.12}$$

where $\dot{\mathcal{H}}^{s-1}(\Gamma_0)$ *is the dual space of* $\dot{\mathcal{H}}^{1-s}(\Gamma_0)$.

By using the continuity of the operator div_{Γ_0} proved in Lemma 3.7, for $\boldsymbol{u} = \sum_{n=1}^{\infty} A_n \boldsymbol{Y}_n + B_n \boldsymbol{V}^{\perp}_n \in \boldsymbol{H}^{-1}_T(\Gamma_0)$, we have then

$$\mathrm{div}_{\Gamma_0} \boldsymbol{u} = \sum_{n=1}^{\infty} \mathrm{div}_{\Gamma_0}(A_n \boldsymbol{Y}_n + B_n \boldsymbol{V}^{\perp}_n) = \sum_{n=1}^{\infty} -\sqrt{\lambda_n} A_n Y_n, \tag{3.13}$$

which converges at least in $\dot{\mathcal{H}}^{-2}(\Gamma_0)$. Similarly, we use Lemma 3.8 to obtain that for $\boldsymbol{u} = \sum_{n=1}^{\infty} A_n \boldsymbol{Y}^{\perp}_n + B_n \boldsymbol{V}_n \in \boldsymbol{H}^{-1}_N(\Gamma_0)$,

$$\mathrm{curl}_{\Gamma_0} \boldsymbol{u} = \sum_{n=1}^{\infty} \mathrm{curl}_{\Gamma_0}(A_n \boldsymbol{Y}^{\perp}_n + B_n \boldsymbol{V}_n) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} A_n Y_n, \tag{3.14}$$

which converges at least in $\dot{\mathcal{H}}^{-2}(\Gamma_0)$.

For $\boldsymbol{u} \in \boldsymbol{H}_b(curl, \widehat{\Omega})$ we define $\gamma_\tau(\boldsymbol{u}) = \boldsymbol{\nu} \times \boldsymbol{u}|_{\Gamma_0}$ and $\pi_\tau(\boldsymbol{u}) = \boldsymbol{\nu} \times (\boldsymbol{u} \times \boldsymbol{\nu})|_{\Gamma_0}$ for the tangential trace and tangential component trace of $\boldsymbol{u}$ on Γ_0, respectively. We also define the spaces of three dimensional vector-valued functions

$$\boldsymbol{H}_T^{-1/2}(\mathrm{div}_{\Gamma_0},\Gamma_0) = \{\boldsymbol{\phi} \in \boldsymbol{H}_T^{-1/2}(\Gamma_0) \;:\; \mathrm{div}_{\Gamma_0}\boldsymbol{\phi} \in \mathcal{H}^{-1/2}(\Gamma_0)\},$$
$$\boldsymbol{H}_N^{-1/2}(\mathrm{curl}_{\Gamma_0},\Gamma_0) = \{\boldsymbol{\phi} \in \boldsymbol{H}_N^{-1/2}(\Gamma_0) \;:\; \mathrm{curl}_{\Gamma_0}\boldsymbol{\phi} \in \mathcal{H}^{-1/2}(\Gamma_0)\}$$

via the natural embedding $\iota : \mathbb{R}^2 \hookrightarrow \{0\} \times \mathbb{R}^2 \subset \mathbb{R}^3$. The space $\boldsymbol{H}_T^{-1/2}(\mathrm{div}_{\Gamma_0},\Gamma_0)$ for tangential traces can be characterized by the norm estimates analyzed in Theorem 3.3 and (3.13): $\boldsymbol{u} \in \boldsymbol{H}_T^{-1/2}(\mathrm{div}_{\Gamma_0},\Gamma_0)$ if and only if $\boldsymbol{u}$ has a series representation $\boldsymbol{u} = \sum_{n=1}^{\infty} A_n\,\boldsymbol{Y}_n + B_n\,\boldsymbol{V}_n^{\perp}$ satisfying

$$\begin{aligned}\|\boldsymbol{u}\|^2_{\boldsymbol{H}_T^{-1/2}(\mathrm{div}_{\Gamma_0},\Gamma_0)} &:= \|\boldsymbol{u}\|^2_{\boldsymbol{H}_T^{-1/2}(\Gamma_0)} + \|\mathrm{div}_{\Gamma_0}\boldsymbol{u}\|^2_{\dot{\mathcal{H}}^{-1/2}(\Gamma_0)}\\ &= \sum_{n=1}^{\infty}(1+\lambda_n)^{1/2}|A_n|^2 + (1+\mu_n)^{-1/2}|B_n|^2 < \infty.\end{aligned}$$

Analogously, due to Theorem 3.6 and (3.14) the space $\boldsymbol{H}_N^{-1/2}(\mathrm{curl}_{\Gamma_0},\Gamma_0)$ for tangential component traces can be interpreted as a space equipped with the norm

$$\begin{aligned}\|\boldsymbol{u}\|^2_{\boldsymbol{H}_N^{-1/2}(\mathrm{curl}_{\Gamma_0},\Gamma_0)} &:= \|\boldsymbol{u}\|^2_{\boldsymbol{H}_N^{-1/2}(\Gamma_0)} + \|\mathrm{curl}_{\Gamma_0}\boldsymbol{u}\|^2_{\dot{\mathcal{H}}^{-1/2}(\Gamma_0)}\\ &= \sum_{n=1}^{\infty}(1+\lambda_n)^{1/2}|A_n|^2 + (1+\mu_n)^{-1/2}|B_n|^2 < \infty\end{aligned}$$

for $\boldsymbol{u} = \sum_{n=1}^{\infty} A_n\,\boldsymbol{Y}_n^{\perp} + B_n\,\boldsymbol{V}_n$.

In order to investigate regularity estimates and continuity of the trace operators, it is required to study liftings of functions in $\boldsymbol{H}_T^{1/2}(\Gamma_0)$ and $\dot{\mathcal{H}}^{3/2}(\Gamma_0)$. The cylindrical geometry of waveguides allows to define liftings as shown in the following lemma.

Lemma 3.9. *For any $\boldsymbol{\phi} \in \boldsymbol{H}_T^{1/2}(\Gamma_0)$ (understood as a vector-valued function in $\mathbb{R}^3$ via the natural embedding ι) there exists $\widetilde{\boldsymbol{\phi}} \in \boldsymbol{H}^1(\widehat{\Omega}) = (\mathcal{H}^1(\widehat{\Omega}))^3$ satisfying $\widetilde{\boldsymbol{\phi}}|_{\Gamma_0} = \boldsymbol{\phi}$ and*

$$\|\widetilde{\boldsymbol{\phi}}\|_{\boldsymbol{H}^1(\widehat{\Omega})} \le C\|\boldsymbol{\phi}\|_{\boldsymbol{H}_T^{1/2}(\Gamma_0)}.$$

Also, for any $\phi \in \dot{\mathcal{H}}^{3/2}(\Gamma_0)$ there exists $\tilde{\phi} \in \mathcal{H}^2(\widehat{\Omega})$ satisfying $\tilde{\phi}|_{\Gamma_0} = \phi$ and

$$\|\tilde{\phi}\|_{\mathcal{H}^s(\widehat{\Omega})} \le C\|\phi\|_{\dot{\mathcal{H}}^{s-1/2}(\Gamma_0)}$$

for $s = 1, 2$.

Proof. Let $\boldsymbol{\phi} \in \boldsymbol{H}_T^{1/2}(\Gamma_0)$ be given by $\boldsymbol{\phi} = \sum_{n=1}^{\infty} A_n\,\boldsymbol{Y}_n + B_n\,\boldsymbol{V}_n^{\perp}$. We denote a semi-infinite cylindrical domain with base Ω by $\Omega_\infty = (-\infty, 0) \times \Omega$. Each cross-section at $x = a$ of Ω_∞ for $a < 0$ can be identified with Ω. Let us define

$$\boldsymbol{\psi}(x,y) = \sum_{n=1}^{\infty}\left(A_n e^{\sqrt{1+\lambda_n}\,x}\,\boldsymbol{Y}_n(y) + B_n e^{\sqrt{1+\mu_n}\,x}\,\boldsymbol{V}_n^{\perp}(y)\right) \quad \text{in } \Omega_\infty.$$

Fubini's theorem enables us to estimate $\boldsymbol{\psi}$ in $\boldsymbol{H}^1(\Omega_\infty)$ as follows,

$$\|\boldsymbol{\psi}\|^2_{\boldsymbol{H}^1(\Omega_\infty)} = \int_{-\infty}^{0} \|\boldsymbol{\psi}(x,\cdot)\|^2_{\boldsymbol{H}^1(\Omega)} + \|\frac{\partial \boldsymbol{\psi}}{\partial x}(x,\cdot)\|^2_{\boldsymbol{L}^2(\Omega)}dx. \tag{3.15}$$

By invoking Theorem 3.3, we obtain that

$$\|\boldsymbol{\psi}(x,\cdot)\|^2_{\boldsymbol{H}^1(\Omega)} = \sum_{n=0}^{\infty}(1+\lambda_n)|A_n|^2 e^{2\sqrt{1+\lambda_n}x} + (1+\mu_n)|B_n|^2 e^{2\sqrt{1+\mu_n}x} \tag{3.16}$$

for $x<0$. For the second term in (3.15) pertaining to the derivative with respect to x, let $\boldsymbol{\psi}_m$ be the partial sum of $\boldsymbol{\psi}$. Noting that for each $x<0$, $\zeta e^{2\zeta x}$ is bounded for all $\zeta = \sqrt{1+\lambda_n}$ or $\sqrt{1+\mu_n}$, $\partial\boldsymbol{\psi}_m/\partial x(x,\cdot)$ converges in $\boldsymbol{L}^2(\Omega)$ for each $x<0$, that is

$$\lim_{m\to\infty}\frac{\partial \boldsymbol{\psi}_m}{\partial x}(x,\cdot) = \sum_{n=1}^{\infty}(1+\lambda_n)^{1/2}A_n e^{\sqrt{1+\lambda_n}x}\,\boldsymbol{Y}_n + (1+\mu_n)^{1/2}B_n e^{\sqrt{1+\mu_n}x}\,\boldsymbol{V}_n^{\perp} \in \boldsymbol{L}^2(\Omega),$$

which implies that

$$\frac{\partial \boldsymbol{\psi}}{\partial x}(x,\cdot) = \sum_{n=1}^{\infty}(1+\lambda_n)^{1/2}A_n e^{\sqrt{1+\lambda_n}x}\,\boldsymbol{Y}_n + (1+\mu_n)^{1/2}B_n e^{\sqrt{1+\mu_n}x}\,\boldsymbol{V}_n^{\perp}$$

and

$$\|\frac{\partial \boldsymbol{\psi}}{\partial x}(x,\cdot)\|^2_{\boldsymbol{L}^2(\Omega)} = \sum_{n=1}^{\infty}(1+\lambda_n)|A_n|^2 e^{2\sqrt{1+\lambda_n}x} + (1+\mu_n)|B_n|^2 e^{2\sqrt{1+\mu_n}x}. \tag{3.17}$$

Now, substitution of (3.16) and (3.17) into (3.15) gives

$$\|\boldsymbol{\psi}\|^2_{\boldsymbol{H}^1(\Omega_\infty)} = \int_{-\infty}^{0}\left(\sum_{n=1}^{\infty}2(1+\lambda_n)|A_n|^2 e^{2\sqrt{1+\lambda_n}x} + 2(1+\mu_n)|B_n|^2 e^{2\sqrt{1+\mu_n}x}\right)dx$$

and the monotone convergence theorem shows that

$$\|\boldsymbol{\psi}\|^2_{\boldsymbol{H}^1(\Omega_\infty)} = \sum_{n=1}^{\infty}(1+\lambda_n)^{1/2}|A_n|^2 + (1+\mu_n)^{1/2}|B_n|^2 = \|\boldsymbol{\phi}\|^2_{\boldsymbol{H}_T^{1/2}(\Gamma_0)}.$$

Finally, by multiplying $\boldsymbol{\psi}$ by a cutoff function χ, which is one for $-L/2<x<0$ and vanishes for $x<-L$, we have a desired lifting $\widetilde{\boldsymbol{\phi}}$, the zero extension of $\chi\boldsymbol{\psi}|_{(-L,0)\times\Omega}$ to $\widehat{\Omega}$, satisfying $\widetilde{\boldsymbol{\phi}}|_{\Gamma_0} = \boldsymbol{\phi}$ and $\|\widetilde{\boldsymbol{\phi}}\|_{\boldsymbol{H}^1(\widehat{\Omega})} \le C\|\boldsymbol{\psi}\|_{\boldsymbol{H}^1(\Omega_\infty)} = C\|\boldsymbol{\phi}\|_{\boldsymbol{H}_T^{1/2}(\Gamma_0)}$, which completes the first part of the lemma.

The second part can be proved in the same way. In this case we take $\psi = \sum_{n=0}^{\infty}A_n e^{\sqrt{1+\lambda_n}x}Y_n(y)$ for $\phi = \sum_{n=0}^{\infty}A_nY_n$ and define $\tilde{\phi}(x,y)$ by the zero extension of $\chi\psi|_{(-L,0)\times\Omega}$ to $\widehat{\Omega}$ with the cutoff function χ defined as above. Then the similar argument used above can show that

$$\begin{aligned}\|\tilde{\phi}\|^2_{\mathcal{H}^1(\widehat{\Omega})} &\le C\|\psi\|^2_{\mathcal{H}^1(\Omega_\infty)} = C\int_{-\infty}^{0}\|\psi(x,\cdot)\|^2_{\mathcal{H}^1(\Omega)} + \|\frac{\partial\psi}{\partial x}(x,\cdot)\|^2_{\mathcal{H}^0(\Omega)}dx\\ &= C\sum_{n=0}^{\infty}(1+\lambda_n)^{1/2}|A_n|^2 = C\|\phi\|^2_{\dot{\mathcal{H}}^{1/2}(\Gamma_0)}\end{aligned}$$

and

$$\begin{aligned}\|\tilde{\phi}\|^2_{\mathcal{H}^2(\widehat{\Omega})} &\leq C\|\psi\|^2_{\mathcal{H}^2(\Omega_\infty)}\\ &= C\int_{-\infty}^{0}\|\psi(x,\cdot)\|^2_{\mathcal{H}^2(\Omega)} + \|\frac{\partial\psi}{\partial x}(x,\cdot)\|^2_{\mathcal{H}^1(\Omega)} + \|\frac{\partial^2\psi}{\partial x^2}(x,\cdot)\|^2_{\mathcal{H}^0(\Omega)}dx\\ &= C\sum_{n=0}^{\infty}\frac{3}{2}(1+\lambda_n)^{3/2}|A_n|^2 \leq C\|\phi\|^2_{\dot{\mathcal{H}}^{3/2}(\Gamma_0)},\end{aligned}$$

which completes the proof. □

The main results of the continuity of the tangential trace and tangential component trace operators will be presented.

Theorem 3.10. *The map* $\gamma_\tau : \boldsymbol{H}_b(curl,\widehat{\Omega}) \to \boldsymbol{H}_T^{-1/2}(\mathrm{div}_{\Gamma_0},\Gamma_0)$ *is continuous.*

Proof. Let $\boldsymbol{u} \in \boldsymbol{H}_b(curl,\widehat{\Omega})$. For $\boldsymbol{\phi} \in \boldsymbol{H}_T^{1/2}(\Gamma_0)$, we denote by $\widetilde{\boldsymbol{\phi}}$ the extension of $\boldsymbol{\phi}$ in $\boldsymbol{H}^1(\widehat{\Omega})$ constructed as in Lemma 3.9. Since $\boldsymbol{\nu}\times\boldsymbol{u} = 0$ on Γ_b, the integration by parts gives

$$(\boldsymbol{\nabla}\times\boldsymbol{u},\widetilde{\boldsymbol{\phi}})_{\widehat{\Omega}} - (\boldsymbol{u},\boldsymbol{\nabla}\times\widetilde{\boldsymbol{\phi}})_{\widehat{\Omega}} = \int_{\Gamma_0}\gamma_\tau(\boldsymbol{u})\cdot\boldsymbol{\phi}\;dy = \langle\gamma_\tau(\boldsymbol{u}),\boldsymbol{\phi}\rangle_{1/2,T,\Gamma_0}. \tag{3.18}$$

Therefore, we have

$$\langle\gamma_\tau(\boldsymbol{u}),\boldsymbol{\phi}\rangle_{1/2,T,\Gamma_0} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}(curl,\widehat{\Omega})}\|\widetilde{\boldsymbol{\phi}}\|_{\boldsymbol{H}^1(\widehat{\Omega})} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}(curl,\widehat{\Omega})}\|\boldsymbol{\phi}\|_{\boldsymbol{H}_T^{1/2}(\Gamma_0)},$$

from which it follows that

$$\|\gamma_\tau(\boldsymbol{u})\|_{\boldsymbol{H}_T^{-1/2}(\Omega)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}(curl,\widehat{\Omega})}. \tag{3.19}$$

For the estimate of $\mathrm{div}_{\Gamma_0}(\gamma_\tau(\boldsymbol{u}))$, let $\phi \in \dot{\mathcal{H}}^{3/2}(\Gamma_0)$ be expressed by the series $\phi = \sum_{n=0}^{\infty}\phi_n Y_n$ and $\tilde{\phi}$ be a lifting given by Lemma 3.9. Then it holds that $\boldsymbol{\nabla}_y\phi = \sum_{n=1}^{\infty}\sqrt{\lambda_n}\phi_n\boldsymbol{Y}_n$ is in $\boldsymbol{H}_T^{1/2}(\Gamma_0)$. Therefore, we can show that

$$(\boldsymbol{\nabla}\times\boldsymbol{u},\boldsymbol{\nabla}\tilde{\phi})_{\widehat{\Omega}} = \int_{\Gamma_0}\gamma_\tau(\boldsymbol{u})\cdot\boldsymbol{\nabla}_y\phi\;dy = \langle\gamma_\tau(\boldsymbol{u}),\boldsymbol{\nabla}_y\phi\rangle_{1/2,T,\Gamma_0}$$

and hence

$$\begin{aligned}\langle\mathrm{div}_{\Gamma_0}(\gamma_\tau(\boldsymbol{u})),\phi\rangle_{3/2,T,\Gamma_0} &= -\langle\gamma_\tau(\boldsymbol{u}),\boldsymbol{\nabla}_y\phi\rangle_{1/2,T,\Gamma_0} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}(curl,\widehat{\Omega})}\|\tilde{\phi}\|_{\mathcal{H}^1(\widehat{\Omega})}\\ &\leq C\|\boldsymbol{u}\|_{\boldsymbol{H}(curl,\widehat{\Omega})}\|\phi\|_{\dot{\mathcal{H}}^{1/2}(\Gamma_0)}.\end{aligned}$$

Since $\dot{\mathcal{H}}^{3/2}(\Gamma_0)$ is dense in $\dot{\mathcal{H}}^{1/2}(\Gamma_0)$, we can conclude that

$$\begin{aligned}\|\mathrm{div}_{\Gamma_0}(\gamma_\tau(\boldsymbol{u}))\|_{\dot{\mathcal{H}}^{-1/2}(\Gamma_0)} &= \sup_{\phi\in\dot{\mathcal{H}}^{3/2}(\Gamma_0)}\frac{|\langle\mathrm{div}_{\Gamma_0}(\gamma_\tau(\boldsymbol{u})),\phi\rangle_{3/2,T,\Gamma_0}|}{\|\phi\|_{\dot{\mathcal{H}}^{1/2}(\Gamma_0)}}\\ &\leq C\|\boldsymbol{u}\|_{\boldsymbol{H}(curl,\widehat{\Omega})}.\end{aligned} \tag{3.20}$$

Finally, combining (3.19) and (3.20) completes the proof. □

Theorem 3.11. *The map* $\pi_\tau : \boldsymbol{H}_b(curl,\widehat{\Omega}) \to \boldsymbol{H}_N^{-1/2}(\mathrm{curl}_{\Gamma_0},\Gamma_0)$ *is continuous.*

Proof. We first note that $\gamma_\tau(\boldsymbol{u}) \in \boldsymbol{H}_T^{-1/2}(\Gamma_0)$ if and only if $\pi_\tau(\boldsymbol{u}) \in \boldsymbol{H}_N^{-1/2}(\Gamma_0)$. In addition, since

$$\begin{aligned}\langle \mathrm{div}_{\Gamma_0}(\gamma_\tau(\boldsymbol{u})), \phi\rangle_{3/2,\Gamma_0} &= -\langle \gamma_\tau(\boldsymbol{u}), \boldsymbol{\nabla}_y \phi\rangle_{1/2,T,\Gamma_0} \\ &= -\langle \pi_\tau(\boldsymbol{u}), \boldsymbol{\nabla}_y^\perp \phi\rangle_{1/2,N,\Gamma_0} = -\langle \mathrm{curl}_{\Gamma_0}(\pi_\tau(\boldsymbol{u})), \phi\rangle_{3/2,\Gamma_0}\end{aligned}$$

for $\phi \in \dot{\mathcal{H}}^{3/2}(\Gamma_0)$, we have $\mathrm{div}_{\Gamma_0}(\gamma_\tau(\boldsymbol{u})) = -\mathrm{curl}_{\Gamma_0}(\pi_\tau(\boldsymbol{u}))$. Therefore, the continuity of the tangential component trace operator π_τ follows immediately from Theorem 3.10. □

4. Appendix

In this appendix we provide the density of $C_n^\infty(\overline{\Omega_E})$ in $\mathcal{H}_n^2(\Omega_E)$ for a bounded and smooth domain $\Omega_E \subset \mathbb{R}^d$.

Lemma 4.1. *The space $C_n^\infty(\overline{\Omega_E})$ is dense in $\mathcal{H}_n^2(\Omega_E)$.*

Proof. For any $u \in \mathcal{H}_n^2(\Omega_E)$, let $g = u|_{\partial\Omega_E}$ be the trace of u on $\partial\Omega_E$. Since $C^\infty(\partial\Omega_E)$ is dense in $\mathcal{H}^{3/2}(\partial\Omega_E)$, there exists a sequence $g_n \in C^\infty(\partial\Omega_E)$ converging to g in $\mathcal{H}^{3/2}(\partial\Omega_E)$. Due to the continuous right inverse of a trace operator, we can find $v_n \in C^\infty(\overline{\Omega_E})$ such that $v_n = g_n$, $\partial v_n/\partial\boldsymbol{\nu} = 0$ on $\partial\Omega_E$ and $\|v_n\|_{\mathcal{H}^2(\Omega_E)} \le C\|g_n\|_{\mathcal{H}^{3/2}(\partial\Omega_E)}$ with C independent of g_n. Since $\{v_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{H}^2(\Omega_E)$, there exists $v \in \mathcal{H}^2(\Omega_E)$ such that $v_n \to v$ in $\mathcal{H}^2(\Omega_E)$. It also satisfies $v = g$ and $\partial v/\partial\boldsymbol{\nu} = 0$ on $\partial\Omega_E$. Now, as $u - v \in \mathcal{H}_0^2(\Omega_E)$, a subspace of functions η in $\mathcal{H}^2(\Omega_E)$ such that $\eta = 0$ and $\partial\eta/\partial\boldsymbol{\nu} = 0$ on $\partial\Omega_E$, the density of $C_0^\infty(\Omega_E)$ in $\mathcal{H}_0^2(\Omega_E)$ guarantees the existence of a sequence ϕ_n in $C_0^\infty(\Omega_E)$ converging to $u - v$ in $\mathcal{H}^2(\Omega_E)$. Finally, we can conclude that $\phi_n + v_n$ in $C_n^\infty(\overline{\Omega_E})$ converges to u in $\mathcal{H}^2(\Omega_E)$, which completes the proof. □

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