Dirichlet-to-Neumann boundary conditions for multiple scattering in waveguides

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Abstract

In this paper, we study a multiple Dirichlet-to-Neumann (MDtN) boundary condition for solving a time-harmonic multiple scattering problem governed by the Helmholtz equation in waveguides that include multiple obstacles, cavities or inhomogeneities with straight waveguides placed between them. The MDtN condition is derived by analyzing analytic solutions represented by Fourier series in the straight waveguides between obstacles, cavities or inhomogeneities. The proposed method is then to remove the straight waveguides between scatterers and impose the MDtN condition on artificial boundaries resulting from domain truncation. This numerical technique can allow a great reduction of computational efforts. The well-posedness of the reduced problem with the full MDtN condition and the reduced problem with truncated MDtN conditions are established. Also the exponential convergence of approximate solutions satisfying truncated MDtN conditions will be proved.

Keywords: multiple Dirichlet-to-Neumann condition, multiple scattering, Helmholtz equation, waveguide

1. Introduction

This paper is concerned with an efficient numerical technique for solving a time-harmonic scattering problem arising in waveguides including multiple obstacles, cavities or inhomogeneities with straight waveguides placed between them. Scattering of electromagnetic and acoustic waves from obstacles, cavities or inhomogeneities takes place in many applications of engineering and science. These scattering problems occur in many different geometric configurations, for instance, scattering problems in exterior domains of bounded obstacles are a main issue for studying scattering problems in open spaces [7] such as radar imaging, or scattering problems for obstacles in a domain bounded by an infinite surface boundary [22, 24], including half-space problems, are an important

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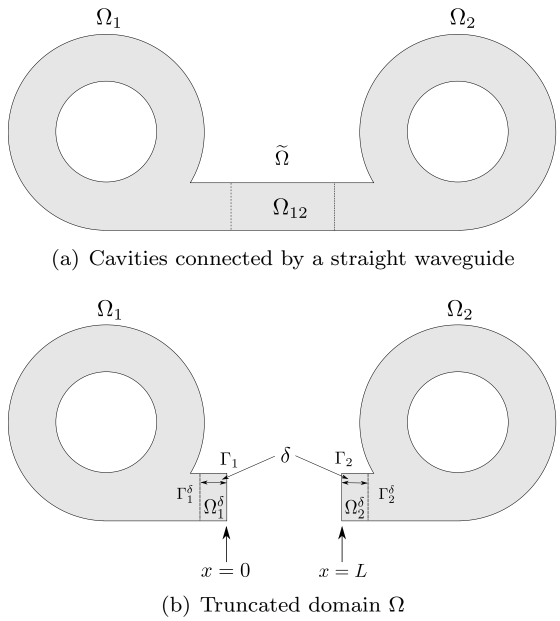


Figure 1: Geometric configuration

subject for research of wave phenomena arising over outdoor grounds or sea surfaces. Scattering problems from cavities embedded in the infinite plane have been studied for e.g., radar cross section [3, 5]. Also, scattering problems in cavities attached by waveguides (which will serve as the model problem in this paper) are investigated for an application such as microwave resonators [6]. For multiple scattering problems when obstacles or cavities are well-separated, there have been intensive studies for efficient numerical methods that can provide accurate approximate solutions with a small amount of computational costs. These are found in many literature for example, [1, 10, 17, 15] for exterior problems, [2] for half-space problems, [18, 25] for cavity problems in half-spaces and [19] for multiple scattering in waveguides among others. See also [20] for an extensive overview of multiple scattering problems. In this paper, we develop an efficient computational method for solving multiple scattering problems in waveguides and give rigorous well-posedness and convergence analyses. Compared with the former method in [19] with boundary conditions based on integral representations, our approach utilizes DtN boundary conditions.

In order to describe our method, we consider a time-harmonic wave scattering problem in a domain $\widetilde{\Omega}$ obtained by joining a finite number of cavities with straight waveguides. For simple presentation, we assume that $\widetilde{\Omega}$ is a bounded and Lipschitz domain in $\mathbb{R}^2$, consisting of two disjoint cavities Ω_1, Ω_2 and a straight waveguide $\Omega_{12} = (0, L) \times \Theta$ connecting two domains Ω_1 and Ω_2 with

Γ_j, $j = 1, 2$ being the common boundary of Ω_j and Ω_{12} (See Figure 1),

$$\widetilde{\Omega} = \Omega_1 \cup \Omega_{12} \cup \Omega_2 \cup \Gamma_1 \cup \Gamma_2.$$

Here Θ is a connected interval in $\mathbb{R}$ and the axis of Ω_{12} is parallel to the x_1-axis, the first coordinate axis of $\mathbb{R} \times \mathbb{R}$. We remark that the method to be developed in this paper can be extended to problems in $\mathbb{R}^3$ as long as eigenpairs of the cross-section $\Theta \subset \mathbb{R}^2$ are available (such as a circular or rectangular domain) to implement DtN boundary conditions defined based on Fourier series as will be seen later. We further assume that the cavity Ω_j, $j = 1, 2$ includes a thin layer Ω_j^δ of width δ, which does not have any inclusion or wave source. Thus any inclusion or wave source of the problem is at least δ away from Γ_j, and the width δ will serve as a parameter of the exponential convergence of the proposed method. The cross-sectional boundaries of Ω_j^δ are denoted by Γ_j^δ and Γ_j as seen Figure 1.

When we are interested in solutions only in the cavities Ω_1 and Ω_2, we may truncate the domain $\widetilde{\Omega}$ to a smaller domain $\Omega := \Omega_1 \cup \Omega_2$ by removing the straight waveguide Ω_{12}. In this procedure, it is required to impose appropriate boundary conditions on artificial boundaries Γ_1 and Γ_2, which can allow us to have solutions coinciding with a restriction of those obtained by solving in the whole domain $\widetilde{\Omega}$. Wave scattering in $\widetilde{\Omega}$ takes place complicatedly since wave fields produced inside the cavities Ω_1 and Ω_2 bounce back and forth through the waveguide Ω_{12}, however the geometric structure of the domain Ω_{12} provides us with simple analytic representations of solutions, one of which is based on Fourier series and the other relies on single and double layer potentials using the Green's function. Both of series and integral representations of solutions in Ω_{12} show that they are decomposed into right-going and left-going components under the time-harmonic assumption $e^{-\mathrm{i}\omega t}$ with angular frequency $\omega > 0$. From this observation we notice that information on traces of right-going components on Γ_1 and that of left-going components on Γ_2 are sufficient to construct the full wave fields on Ω_{12}. By examining the DtN operators for right-going and left-going components, we can derive the MDtN condition for solutions on Γ_j. Therefore we remove Ω_{12} and introduce instead new auxiliary variables representing the traces of right-going and left-going components only on artificial boundaries and satisfying the MDtN condition, which allows a drastic reduction of computational efforts. A remark on the analysis is that series representations play an important role in deriving the MDtN condition whereas layer potentials given by integral representations are the main ingredient in the stability analysis. In fact, the layer potential theory (see e.g., [7, 23]) enables us to have a stable decomposition of the right-going and left-going components of scattering solutions in Ω_{12}.

The use of the MDtN condition was initially proposed for exterior multiple scattering problems in [10]. Here we apply the idea to the multiple scattering problems arising in waveguides and establish (1) well-posedness of the reduced problem supplemented with the MDtN condition on artificial boundaries and an equivalence of the reduced problem to the original full problem, (2) well-posedness of the approximate problem obtained by replacing the infinite series of the MDtN condition with a finite series with M numbers of terms for sufficiently large M, and (3) exponential convergence of approximate solutions obtained by the proposed method as M tends toward infinity.

The paper is organized as follows. In Section 2 we introduce the MDtN condition and the reduced problem posed only on Ω with the MDtN condition. Also, the well-posedness of the reduced problem and the equivalence between the reduced problem and the original problem is investigated. In Section 3 we propose a numerical technique by truncating the infinite series of the MDtN condition. There we study the well-posedness of the reduced problem supplemented with the truncated MDtN condition and the exponential convergence of approximate solutions. Finally, numerical experiments illustrating the convergence theory will be presented in Section 4.

2. Multiple scattering and reduced problem

2.1. Multiple Dirichlet-to-Neumann condition

The model problem under consideration is the Helmholtz equation

$$\begin{aligned} -\Delta u^{ex} - k^2 u^{ex} &= f \quad \text{in } \widetilde{\Omega}, \\ \frac{\partial u^{ex}}{\partial \nu_{\widetilde{\Omega}}} &= 0 \quad \text{on } \partial\widetilde{\Omega}, \end{aligned} \tag{2.1}$$

where k is a positive wavenumber and $f \in L^2(\widetilde{\Omega})$ is a source term supported in $\Omega = \Omega_1 \cup \Omega_2$ away from the thin layers $\Omega_1^\delta \cup \Omega_2^\delta$. Here $\nu_{\mathcal{D}}$ represents the unit normal vector on $\partial\mathcal{D}$ pointing outward from a domain $\mathcal{D}$. From here on we use $(\cdot,\cdot)_{\mathcal{D}}$ for L^2-inner product over the complex field $\mathbb{C}$ in a domain $\mathcal{D}$

$$(u, v)_{\mathcal{D}} = \int_{\mathcal{D}} u(x)\bar{v}(x)dx,$$

and we denote the dual space (as the space of anti-linear functionals) of $H^1(\mathcal{D})$ by $\widetilde{H}^{-1}(\mathcal{D})$.

We consider a weak solution $u^{ex} \in H^1(\widetilde{\Omega})$ satisfying

$$\mathcal{A}(u^{ex}, \phi) = (f, \phi)_\Omega \text{ for } \phi \in H^1(\widetilde{\Omega}), \tag{2.2}$$

where

$$\mathcal{A}(u, \phi) = (\nabla u, \nabla \phi)_{\widetilde{\Omega}} - k^2 (u, \phi)_{\widetilde{\Omega}}.$$

For unique solvability of the problem (2.2), we assume that k^2 is not a Neumann eigenvalue of $-\Delta$ in $\widetilde{\Omega}$. Then the Fredholm alternative theorem implies that there exists a unique solution $u^{ex} \in H^1(\widetilde{\Omega})$ to the problem (2.2) satisfying

$$\|u^{ex}\|_{H^1(\widetilde{\Omega})} \leq C\|f\|_{L^2(\Omega)}. \tag{2.3}$$

Throughout the paper we will use C for a generic constant that depend only on the domain $\widetilde{\Omega}$ and wavenumber k.

Denoting the second coordinate variable for $\mathbb{R} \times \mathbb{R}$ by x_2, let $\{Y_n\}_{n=0}^{\infty}$ be the set of an orthonormal basis in $L^2(\Theta)$ consisting of Neumann eigenfunctions of the negative cross-sectional Laplace operator $-\Delta_{x_2}$ on Θ, associated with eigenvalues λ_n,

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots .$$

Due to the orthonormality of Y_n, fractional Sobolev spaces $H^s(\Theta)$ for $-1/2 \leq s \leq 1/2$ is characterized by the space of functions $\phi = \sum_{n=0}^{\infty} \phi_n Y_n$ satisfying, (see [16]),

$$\|\phi\|_{H^s(\Theta)}^2 = \sum_{n=0}^{\infty} (1 + \lambda_n^2)^s |\phi_n|^2 < \infty.$$

For any $M \in \mathbb{N}$, we define $H^s_{\leq M}(\Theta)$ by the subspace spanned by $\{Y_n\}_{n=0}^{M}$ in $H^s(\Theta)$. Similarly, $H^s_{=M}(\Theta)$, $H^s_{\neq M}(\Theta)$ and $H^s_{>M}(\Theta)$ can be defined accordingly to the symbols used in the subscript.

Let $\mu_n^2 = k^2 - \lambda_n^2$. Since λ_n approaches infinity as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $\lambda_n \leq k$ for $n \leq N$ and $\lambda_n > k$ for $n > N$. Thus, $\mu_n = \sqrt{k^2 - \lambda_n^2} \geq 0$ for $n \leq N$ and $\mu_n = \mathrm{i}\tilde{\mu}_n$ with $\tilde{\mu}_n = \sqrt{\lambda_n^2 - k^2} > 0$ for $n > N$ by taking a negative real axis branch cut. In a certain case, we may have $\mu_N = 0$ and such a mode corresponding to $n = N$ is called a cutoff mode. By separation of variables, it can be shown that the solution u^{ex} in Ω_{12} is a superposition of infinitely many different modes u_n^{ex} for $n = 0, 1, \dots,$

$$u_n^{ex}(x_1, x_2) = \begin{cases} (A_n e^{\mathrm{i}\mu_n x_1} + B_n e^{-\mathrm{i}\mu_n x_1}) Y_n(x_2) & \text{if } \mu_n \neq 0, \\ (A_n + B_n x_1) Y_n(x_2) & \text{if } \mu_n = 0. \end{cases} \tag{2.4}$$

Since the Fourier coefficient (linear function of x_1) of general solutions of cutoff modes is different from that (exponential function of x_1) of other modes, we assume that cutoff modes are involved in the problem for a complete analysis and reserve the index N for cutoff modes.

From the solution formula (2.4), we note that the n-th mode of the solution u^{ex} for $\mu_n \neq 0$ is decomposed into the right-going and left-going components under the time-harmonic assumption $e^{-\mathrm{i}\omega t}$ with angular frequency $\omega > 0$. For $n = N$, the cutoff mode can also be decomposed into two parts by expressing $A_N + B_N x_1$ in terms of linear Lagrange basis polynomials $\mathbb{L}_0$ and $\mathbb{L}_L$ with two nodes 0 and L on the interval $(0, L)$,

$$u_N^{ex}(x_1, x_2) = (u_N^{ex}|_{\Gamma_1} \mathbb{L}_0(x_1) + u_N^{ex}|_{\Gamma_2} \mathbb{L}_L(x_1)) Y_N(x_2).$$

Now, we denote the superposition of all right-going (left-going) components including the decreasing component of cutoff modes in the propagating direction by u^{right} (u^{left}, respectively), i.e.,

$$u^{\text{right}}(x_1, \cdot) = u_N^{ex}|_{\Gamma_1} \mathbb{L}_0(x_1) Y_N + \sum_{n \neq N} A_n e^{\mathrm{i}\mu_n x_1} Y_n, \tag{2.5}$$

$$u^{\text{left}}(x_1, \cdot) = u_N^{ex}|_{\Gamma_2} \mathbb{L}_L(x_1) Y_N + \sum_{n \neq N} B_n e^{-\mathrm{i}\mu_n x_1} Y_n, \tag{2.6}$$

which allows us to have a decomposition of the solution u^{ex} in Ω_{12}

$$u^{ex} = u^{\mathrm{right}} + u^{\mathrm{left}}. \tag{2.7}$$

Remark 2.1. *According to the decomposition* (2.7) *of the solution* u^{ex}, *we interpret* u^{right} *and* u^{left} *as outgoing components of* u^{ex} *from* Ω_1 *and* Ω_2 *respectively. Similarly, they are thought of as incoming components into* Ω_2 *and* Ω_1, *respectively.*

In order to obtain a reduced problem posed only on Ω by removing the straight waveguide Ω_{12} from $\widetilde{\Omega}$, it is required to understand Dirichlet and Neumann traces of solutions on Γ_1 and Γ_2. To this end, we introduce the following Dirichlet-to-Neumann (DtN) and Dirichlet-to-Dirichlet (DtD) operators. The DtN and DtD operators for exterior problems can be found in [10].

1. For the Neumann trace of outgoing components from the domain Ω_i we introduce the well-known DtN map $T_{ii} : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ defined by

$$T_{ii}(\phi) = -\frac{\phi_N}{L} Y_N + \sum_{n \neq N} \mathrm{i}\mu_n \phi_n Y_n$$

   for $\phi = \sum_{n=0}^{\infty} \phi_n Y_n$ in $H^{1/2}(\Gamma_i)$.
2. For the Neumann trace of incoming components into the domain Ω_j, we define a transferred DtN map $T_{ij} : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_j)$, $i \neq j$, by

$$T_{ij}(\phi) = \frac{\phi_N}{L} Y_N + \sum_{n \neq N} -\mathrm{i}\mu_n e^{\mathrm{i}\mu_n L} \phi_n Y_n$$

   for $\phi = \sum_{n=0}^{\infty} \phi_n Y_n$ in $H^{1/2}(\Gamma_i)$.
3. At last, for the Dirichlet trace of incoming components into the domain Ω_j we define a transferred DtD map $P_{ij} : H^{1/2}(\Gamma_i) \to H^{1/2}(\Gamma_j)$, $i \neq j$, by

$$P_{ij}(\phi) = \sum_{n \neq N} e^{\mathrm{i}\mu_n L} \phi_n Y_n$$

   for $\phi = \sum_{n=0}^{\infty} \phi_n Y_n$ in $H^{1/2}(\Gamma_i)$.

Remark 2.2. *These operators* T_{ij} *and* P_{ij} *can be defined as continuous operators* $T_{ij} : H^s(\Gamma_i) \to H^{s-1}(\Gamma_j)$ *and* $P_{ij} : H^s(\Gamma_i) \to H^s(\Gamma_j)$ *for* $s \in \mathbb{R}$ *as well.*

In fact, $T_{ij}(\phi)$ and $P_{ij}(\phi)$ for $\phi \in H^{1/2}(\Gamma_i)$ and $i \neq j$ are associated with the Neumann and the Dirichlet traces on Γ_j of an outgoing solution in the straight waveguide Ω_{12} in the sense of Remark 2.1 with a Dirichlet data given by ϕ on Γ_i, respectively. More precisely,

$$T_{11}(\phi) = \frac{\partial u^{\mathrm{right}}}{\partial \nu_{\Omega_1}}|_{\Gamma_1} \quad \text{and} \quad T_{12}(\phi) = \frac{\partial u^{\mathrm{right}}}{\partial \nu_{\Omega_2}}|_{\Gamma_2}$$

with $u^{\text{right}} = \phi$ on Γ_1, and

$$T_{21}(\phi) = \frac{\partial u^{\text{left}}}{\partial \nu_{\Omega_1}}|_{\Gamma_1} \quad \text{and} \quad T_{22}(\phi) = \frac{\partial u^{\text{left}}}{\partial \nu_{\Omega_2}}|_{\Gamma_2}$$

with $u^{\text{left}} = \phi$ on Γ_2. Similarly, $P_{ij}(\phi)$ is the operator defined as

$$\begin{aligned} P_{12}(\phi) &= u^{\text{right}}|_{\Gamma_2} \text{ with } u^{\text{right}} = \phi \quad \text{on } \Gamma_1, \\ P_{21}(\phi) &= u^{\text{left}}|_{\Gamma_1} \text{ with } u^{\text{left}} = \phi \quad \text{on } \Gamma_2. \end{aligned}$$

Since it holds that

$$\frac{|\mu_n|^2}{1+\lambda_n^2} = \frac{|k^2 - \lambda_n^2|}{1+\lambda_n^2} < C \text{ for } n = 0, 1, \ldots \tag{2.8}$$

with C depending only on k, we have the continuity of T_{ij} and P_{ij}. In addition, we need a lower bound of $|\mu_n|$ for non-cutoff mode for norm estimates associated with layer potentials afterward. For this we denote the smallest non-zero $|\mu_n|$ by $\mu_{\min} = \min\{|\mu_n| \,:\, \mu_n \neq 0\}$, which depends on the position of k with respect to the distribution of λ_n. In particular, a careful analysis is required in case that k does not coincide with any of λ_n but k is close to one of λ_n. The mode corresponding to such λ_n is called a near-cutoff mode, that is, $|\mu_n| = \mu_{\min} \ll 1$. In order to handle both cutoff modes and near-cutoff modes (they do not exist simultaneously though), we keep the index N for cutoff modes and assume that $\mu_{\min} \ll 1$ for near-cutoff modes. Note that if $k \neq \lambda_n$ for all n, the index for near-cutoff modes is $N-1$ or $N+1$ by the assumption that $k < \lambda_{N+1}$ and the notational convention with N reserved for cutoff modes. For showing the influence of $\mu_{\min}$ on the stability and convergence of the proposed method, norm estimates will be made with constants involving $\mu_{\min}$. However if near-cutoff modes do not exist, we can ignore the dependence on $\mu_{\min}$.

2.2. Reduced problem supplemented with the MDtN condition

In this subsection we propose a domain truncation method to solve the problem (2.2) by removing Ω_{12} from $\widetilde{\Omega}$. To this end let $W = H^1(\Omega) \times H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_2)$ equipped with the norm,

$$\|(u, u_1, u_2)\|_W = (\|u\|^2_{H^1(\Omega)} + \mu^2_{\min}(\|u_1\|^2_{H^{1/2}(\Gamma_1)} + \|u_2\|^2_{H^{1/2}(\Gamma_2)}))^{1/2}$$

for $(u, u_1, u_2) \in W$, and we denote a complement of $\bar{\Gamma}_1 \cup \bar{\Gamma}_2$ from $\partial\Omega$ by $\Gamma = \partial\Omega \setminus (\bar{\Gamma}_1 \cup \bar{\Gamma}_2)$.

The reduced problem resulting from removing Ω_{12} can be written as a problem to find $(u, u_1, u_2) \in W$ satisfying

$$\begin{aligned} \Delta u + k^2 u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma \end{aligned} \tag{2.9}$$

and

$$\frac{\partial u}{\partial \nu} = T_{11}(u_1) + T_{21}(u_2) \quad \text{on } \Gamma_1, \quad \frac{\partial u}{\partial \nu} = T_{12}(u_1) + T_{22}(u_2) \quad \text{on } \Gamma_2, \tag{2.10}$$

$$u = u_1 + P_{21}(u_2) \quad \text{on } \Gamma_1, \qquad u = P_{12}(u_1) + u_2 \quad \text{on } \Gamma_2. \tag{2.11}$$

Theorem 2.3. *There exists a unique solution (u, u_1, u_2) in W to the problem (2.9)-(2.11). In addition, if u^{ex} is the solution to the problem (2.2), then u coincides with the restriction of u^{ex} to Ω.*

PROOF. Let u^{ex} be the unique solution to the problem (2.2). By the definition of T_{ij} and P_{ij} based on the decomposition (2.7) of the solution u^{ex} in Ω_{12}, it is obvious that the triple $(u, u_1, u_2) \in W$ defined by $u := u^{ex}|_{\Omega}$, $u_1 := u^{\text{right}}|_{\Gamma_1}$ and $u_2 := u^{\text{left}}|_{\Gamma_2}$ solves the the reduced problem, which asserts the existence of a solution to the problem (2.9)-(2.11).

Conversely, suppose that $(u, u_1, u_2) \in W$ is a solution to the problem (2.9)-(2.11). Let w_1 and $w_2 \in H^1(\Omega_{12})$ be right-going and left-going solutions of the form (2.5) and (2.6) determined by the Dirichlet data $w_1 = u_1$ on Γ_1 and $w_2 = u_2$ on Γ_2, respectively. We then define $\hat{u}$ by

$$\hat{u} = \begin{cases} u & \text{in } \Omega, \\ w_1 + w_2 & \text{in } \Omega_{12}. \end{cases}$$

We claim that $\hat{u} = u^{ex}$ in $\widetilde{\Omega}$. Indeed, by the definition of T_{ij} and P_{ij}, it is obvious that

$$P_{ij}(u_i) = w_i|_{\Gamma_j} \quad \text{and} \quad T_{ij}(u_i) = \frac{\partial w_i}{\partial \nu_{\Omega_j}}|_{\Gamma_j},$$

which implies that

$$u = w_1 + w_2 \quad \text{and} \quad \frac{\partial u}{\partial \nu_\Omega} = \frac{\partial}{\partial \nu_\Omega}(w_1 + w_2) \quad \text{on } \Gamma_1 \cup \Gamma_2.$$

Therefore it can be shown that $\hat{u}$ is in $H^1(\Omega)$ and solves the problem (2.2). Finally, since the problem (2.2) has at most one solution, $\hat{u}$ needs to coincide with u^{ex} and the uniqueness of solutions to the problem (2.9)-(2.11) follows. □

As seen in Theorem 2.3, the problem (2.9)-(2.11) has a unique solution $(u, u_1, u_2) \in W$. Since $u^{ex}|_{\Omega} = u$, the stability (2.3) of the problem (2.2) yields that

$$\|u\|_{H^1(\Omega)} \le \|u^{ex}\|_{H^1(\widetilde{\Omega})} \le C\|f\|_{L^2(\Omega)}. \tag{2.12}$$

In order to estimate u_1 and u_2, which are the Dirichlet traces of u^{right} and u^{left}, respectively, in terms of the wave source f, we consider an orthogonal decomposition of functions in $H^1(\Omega_{12})$. Assuming that the N-th mode is assigned to cutoff modes, any function $v \in H^1(\Omega_{12})$ can be written uniquely as

$$v = v_N + v_{\neq N}, \tag{2.13}$$

where

$$v_N(x_1,x_2) = (v(x_1,\cdot),Y_N)_\Theta Y_N(x_2),$$
$$v_{\neq N}(x_1,x_2) = \sum_{n\neq N}(v(x_1,\cdot),Y_n)_\Theta Y_n(x_2).$$

By Fubini's theorem, one can easily show that

$$\|v\|^2_{H^1(\Omega_{12})} = \|v_N\|^2_{H^1(\Omega_{12})} + \|v_{\neq N}\|^2_{H^1(\Omega_{12})}$$

and it gives the orthogonal decomposition of $H^1(\Omega_{12})$

$$H^1(\Omega_{12}) = H^1_{=N}(\Omega_{12}) \oplus H^1_{\neq N}(\Omega_{12})$$

according to (2.13). Thus we will do norm estimates by splitting functions in $H^1(\Omega_{12})$ into cutoff components and non-cutoff components. Now, u^{right} and u^{left} are broken into cutoff components and non-cutoff components, that is, $u^{\text{right}} = u^{\text{right}}_N + u^{\text{right}}_{\neq N}$ and $u^{\text{left}} = u^{\text{left}}_N + u^{\text{left}}_{\neq N}$, and we investigate each component of u^{right} and u^{left} in the next subsections in more details.

2.3. Estimates of non-cutoff components: Integral representation and layer potentials

This subsection is devoted to estimating non-cutoff components $u^{\text{right}}_{\neq N}$ and $u^{\text{left}}_{\neq N}$ of u^{right} and u^{left}, respectively, by employing the theory of single and double layer potentials. The estimates obtained in this subsection will be used for the stability estimate of solutions to the problem (2.9)-(2.11).

For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, let $G(x, y)$ be the Green's function of the Helmholtz equation in the infinite waveguide $\Omega_\infty = \mathbb{R} \times \Theta$ excluding cutoff components,

$$G(x,y) = \sum_{n\neq N} Y_n(x_2)Y_n(y_2)\frac{e^{\mathrm{i}\mu_n|x_1-y_1|}}{-2\mathrm{i}\mu_n} \tag{2.14}$$

(see e.g., [6, 8]). We define the single and double layer potentials on Γ_j, $j = 1, 2$

$$\mathbb{S}_j(\psi) = \int_{\Gamma_j} G(\cdot,y)\psi(y)dy_2 \quad \text{for } \psi \in H^{-1/2}(\Gamma_j),$$
$$\mathbb{D}_j(\psi) = \int_{\Gamma_j} \frac{\partial G}{\partial \nu_{\Omega_{12}}}(\cdot,y)\psi(y)dy_2 \quad \text{for } \psi \in H^{1/2}(\Gamma_j).$$

Then the non-cutoff component $u^{ex}_{\neq N}$ of the solution u^{ex} in Ω_{12} can be written as

$$u^{ex}_{\neq N}(x) = \int_{\Gamma_1\cup\Gamma_2}\left[\frac{\partial u^{ex}}{\partial \nu_{\Omega_{12}}}(y)G(x,y) - \frac{\partial G}{\partial \nu_{\Omega_{12}}}(x,y)u^{ex}(y)\right]dy_2.$$

Due to the definition of the Green's function (2.14), it turns out that

$$\begin{aligned} u^{\text{right}}_{\neq N}(x) &= \int_{\Gamma_1}\left[\frac{\partial u^{ex}}{\partial \nu_{\Omega_{12}}}(y)G(x,y) - \frac{\partial G}{\partial \nu_{\Omega_{12}}}(x,y)u^{ex}(y)\right]dy_2, \\ u^{\text{left}}_{\neq N}(x) &= \int_{\Gamma_2}\left[\frac{\partial u^{ex}}{\partial \nu_{\Omega_{12}}}(y)G(x,y) - \frac{\partial G}{\partial \nu_{\Omega_{12}}}(x,y)u^{ex}(y)\right]dy_2, \end{aligned} \tag{2.15}$$

respectively. Here we note that although u^{right} and u^{left} are components of u^{ex} defined only on Ω_{12}, u^{right} can be extended to any $x > 0$ according to the formulas (2.15), and so can u^{left} for any $x < L$ analogously.

In order to estimate the single and double layer potentials we let

$$\Omega_1^* = (0,1) \times \Theta \quad \text{and} \quad \Omega_2^* = (L-1, L) \times \Theta,$$

which are not necessarily contained in Ω_{12} as mentioned above, but the left boundary $\{0\} \times \Theta$ of Ω_1^* is Γ_1 and the right boundary $\{L\} \times \Theta$ of Ω_2^* is Γ_2. We will first consider the Newton potential on Ω_j^*,

$$\mathbb{N}_j(\phi) = \iint_{\Omega_j^*} G(\cdot, y)\phi(y)dy$$

for $\phi \in \widetilde{H}^{-1}(\Omega_j^*)$. We note that $H^1(\Omega_\infty)$ and $\widetilde{H}^{-1}(\Omega_\infty)$ are Sobolev spaces of functions $v(x_1, x_2) = \sum_{n=0}^\infty v_n(x_1)Y_n(x_2)$ satisfying

$$\|v\|^2_{H^s(\Omega_\infty)} \text{ or } \|v\|^2_{\widetilde{H}^s(\Omega_\infty)} = \sum_{n=0}^\infty \int_{\mathbb{R}} (1 + \lambda_n^2 + |\xi_1|^2)^s |\hat{v}_n(\xi_1)|^2 d\xi_1 < \infty,$$

with $s = 1, -1$, respectively, where $\hat{v}_n$ is the Fourier transform of v_n with the Fourier variable ξ_1 for x_1. We define $H^1_{\neq N}(\Omega_\infty)$ and $H^1_{\neq N}(\Omega_j^*)$ analogously to $H^1_{\neq N}(\Omega_{12})$.

Lemma 2.4. *The Newton potential $\mathbb{N}_j : \widetilde{H}^{-1}(\Omega_j^*) \to H^1_{\neq N}(\Omega_j^*)$ is continuous,*

$$\|\mathbb{N}_j(\phi)\|_{H^1(\Omega_j^*)} \le \frac{C}{\mu_{\min}} \|\phi\|_{\widetilde{H}^{-1}(\Omega_j^*)} \quad \textit{for } \phi \in \widetilde{H}^{-1}(\Omega_j^*).$$

If a near-cutoff mode does not exist in ϕ, then $\mu_{\min}$ is not involved.

PROOF. We only prove the case of $j = 1$ as the other case is proved in the same way. Let $\phi \in C_0^\infty(\Omega_1^*)$. Then the zero extension $\tilde{\phi}$ of ϕ to Ω_∞ has a series representation in Ω_∞,

$$\tilde{\phi}(x_1, x_2) = \sum_{n=0}^\infty \tilde{\phi}_n(x_1)Y_n(x_2),$$

where $\tilde{\phi}_n(x_1) = (\tilde{\phi}(x_1, \cdot), Y_n)_\Theta$ and they are supported in the interval $(0, 1)$. In addition, $v = \mathbb{N}_j(\phi)$ can be written as a Fourier series,

$$v(x_1, x_2) = \sum_{n=0}^\infty v_n(x_1)Y_n(x_2),$$

where v_n is given by

$$\begin{aligned} v_n(x_1) &= (v(x_1, \cdot), Y_n)_\Theta \\ &= \iint_{\Omega_1^*} Y_n(y_2) \frac{e^{\mathrm{i}\mu_n|x_1-y_1|}}{-2\mathrm{i}\mu_n} \tilde{\phi}(y_1, y_2)dy = \int_{\mathbb{R}} \frac{e^{\mathrm{i}\mu_n|x_1-y_1|}}{-2\mathrm{i}\mu_n} \tilde{\phi}_n(y_1)dy_1 \end{aligned}$$

for $n \neq N$ and $v_N = 0$ for $n = N$ as the Green's function G does not have the N-th mode. Now, we shall estimate v_n in two cases, one for propagating modes and possibly near-cutoff mode, $n \leq N+1$, $n \neq N$, and the other for all other evanescent modes, $n > N+1$.

If $n > N+1$, then $\mu_n = \mathrm{i}\tilde{\mu}_n$ with $\tilde{\mu}_n > 0$. Since there is no near-cutoff mode in this case, we have

$$\frac{1+\lambda_n^2}{\tilde{\mu}_n^2} \leq C. \tag{2.16}$$

Noting that

$$g_n(x_1) = \frac{e^{\mathrm{i}\mu_n|x_1|}}{-2\mathrm{i}\mu_n}$$

is the Green's function to the Helmholtz equation with wavenumber μ_n in $\mathbb{R}$ satisfying the radiation condition at infinity, we see that v_n is the solution to the Helmholtz equation in $\mathbb{R}$,

$$\frac{d^2 v_n}{dx_1^2} + \mu_n^2 v_n = -\tilde{\phi}_n$$

with the radiation condition at infinity. Denoting the Fourier transforms of v_n and $\tilde{\phi}_n$ by $\hat{v}_n$ and $\hat{\phi}_n$, respectively, we can obtain that

$$(\xi_1^2 + \tilde{\mu}_n^2)\hat{v}_n(\xi_1) = \hat{\phi}_n(\xi_1),$$

from which together with (2.16) it follows that

$$(1+\lambda_n^2+\xi_1^2)|\hat{v}_n(\xi_1)| \leq C|\hat{\phi}_n(\xi_1)|. \tag{2.17}$$

In case that $n \leq N+1$, $n \neq N$, we introduce a cutoff function χ of $x_1 \geq 0$ such that

$$\chi(x_1) = 1 \text{ for } 0 \leq x_1 \leq 1, \qquad \chi(x_1) = 0 \text{ for } x_1 \geq 2,$$
$$\|\chi\|_{L^\infty(\mathbb{R}_+)} \leq 1, \quad \|\frac{d\chi}{dx_1}\|_{L^\infty(\mathbb{R}_+)} \leq C, \quad \|\frac{d^2\chi}{dx_1^2}\|_{L^\infty(\mathbb{R}_+)} \leq C,$$

and define

$$v_{n,\chi}(x_1) := \int_0^1 \chi(|x_1-y_1|)\frac{e^{\mathrm{i}\mu_n|x_1-y_1|}}{-2\mathrm{i}\mu_n}\tilde{\phi}_n(y_1)dy_1.$$

Since $v_{n,\chi}$ satisfies $v_{n,\chi}(x_1) = v_n(x_1)$ for $0 \leq x_1 \leq 1$ and has compact support, it holds that

$$\|\frac{d^\ell v_n}{dx_1^\ell}\|_{L^2((0,1))} = \|\frac{d^\ell v_{n,\chi}}{dx_1^\ell}\|_{L^2((0,1))} \leq \|\frac{d^\ell v_{n,\chi}}{dx_1^\ell}\|_{H^1(\mathbb{R})} \tag{2.18}$$

for $\ell = 0, 1$. Noting that $\tilde{\phi}_n$ is supported in $(0,1)$, the Fourier transform of $v_{n,\chi}$ with a change of variables $z = x_1 - y_1$ for x_1 can be expressed as

$$\begin{aligned}\hat{v}_{n,\chi}(\xi_1) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi(|x_1-y_1|)\frac{e^{\mathrm{i}\mu_n|x_1-y_1|}e^{-\mathrm{i}x_1\xi_1}}{-2\mathrm{i}\mu_n}\tilde{\phi}_n(y_1)dy_1dx_1 \\ &= \left(\int_{\mathbb{R}} \chi(|z|)\frac{e^{\mathrm{i}(\mu_n|z|-\xi_1 z)}}{-2\mathrm{i}\mu_n}dz\right)\hat{\phi}_n(\xi_1) := J_n(\xi_1)\hat{\phi}_n(\xi_1).\end{aligned} \tag{2.19}$$

Here J_n can be rewritten as

$$J_n(\xi_1) = \int_0^\infty \chi(|z|)\frac{e^{\mathrm{i}\mu_n z}}{-\mathrm{i}\mu_n}\cos(\xi_1 z)dz.$$

One can show by a simple computation (or see e.g., [21, Lemma 3.7]) and the inequalities $\mu_{\min} \le |\mu_n| \le Ck$ that

$$|J_n(\xi_1)| \le \frac{C}{|\mu_n|} \le \frac{C}{\mu_{\min}} \quad \text{and} \quad |\xi_1|^2|J_n(\xi_1)| \le C(|\mu_n| + \frac{1}{|\mu_n|}) \le \frac{C}{\mu_{\min}},$$

from which it follows that

$$(1+\lambda_n^2+|\xi_1|^2)|\hat{v}_{n,\chi}| = ((1+\lambda_n^2)|J_n| + |\xi_1|^2|J_n|)|\hat{\phi}_n| \le \frac{C}{\mu_{\min}}|\hat{\phi}_n| \tag{2.20}$$

due to the fact that $0 \le \lambda_n < Ck$ for $0 \le n \le N+1$, $n \ne N$.

Therefore by combining (2.17), (2.18), (2.20) and the Plancherel theorem, we obtain that

$$\begin{aligned}
\|v\|^2_{H^1(\Omega_1^*)} &= \sum_{n=0}^{\infty}\left[(1+\lambda_n^2)\|v_n\|^2_{L^2((0,1))} + \|\frac{dv_n}{dx_1}\|^2_{L^2((0,1))}\right] \\
&\le \frac{C}{\mu_{\min}^2}\sum_{n=0}^{\infty}\int_{\mathbb{R}}(1+\lambda_n^2+|\xi_1|^2)^{-1}|\hat{\phi}_n(\xi_1)|^2 d\xi_1 \\
&= \frac{C}{\mu_{\min}^2}\|\tilde{\phi}\|^2_{\widetilde{H}^{-1}(\Omega_\infty)} \le \frac{C}{\mu_{\min}^2}\|\phi\|^2_{\widetilde{H}^{-1}(\Omega_1^*)},
\end{aligned}$$

which proves the continuity of the Newton potential by using a density argument. Finally, as $v_N = 0$, v belongs to $H^1_{\ne N}(\Omega_1^*)$, which completes the proof. □

Now, we can show that the trace of the normal derivative of the Newton potential on Γ_j is continuous.

Lemma 2.5. *It holds that*

$$\frac{\partial}{\partial\nu_{\Omega_j^*}}\mathbb{N}_j : \widetilde{H}^{-1}(\Omega_j^*) \to H^{-1/2}_{\ne N}(\Gamma_j)$$

is continuous,

$$\|\frac{\partial}{\partial\nu_{\Omega_j^*}}\mathbb{N}_j(\phi)\|_{H^{-1/2}(\Gamma_j)} \le \frac{C}{\mu_{\min}}\|\phi\|_{\widetilde{H}^{-1}(\Omega_j^*)} \quad \textit{for } \phi \in \widetilde{H}^{-1}(\Omega_j^*).$$

If a near-cutoff mode does not exist in ϕ, then $\mu_{\min}$ is not involved.

PROOF. We will prove only the case of $j=1$ and the other case can be proved in the same way. We first note that for $w \in H^{1/2}(\Gamma_1)$ there exists an extension $\tilde{w} \in H^1(\Omega_1^*)$ of w such that $w = \tilde{w}$ on Γ_1 and $\tilde{w} = 0$ on $\{1\}\times\Theta$ and satisfying

$$\|\tilde{w}\|_{H^1(\Omega_1^*)} \le C\|w\|_{H^{1/2}(\Gamma_1)}.$$

Since $\mathbb{N}_1(\phi)$ for $\phi \in \widetilde{H}^{-1}(\Omega_1^*)$ solves the Helmholtz equation

$$(-\Delta - k^2)\mathbb{N}_1(\phi) = \phi \quad \text{in } \Omega_1^*,$$

integration by parts leads to

$$\langle \frac{\partial}{\partial \nu_{\Omega_1^*}} \mathbb{N}_1(\phi), w\rangle_{1/2,\Gamma_1} = (\nabla \mathbb{N}_1(\phi), \nabla \tilde{w})_{\Omega_1^*} - k^2(\mathbb{N}_1(\phi), \tilde{w})_{\Omega_1^*} - \langle \phi, \tilde{w}\rangle_{1,\Omega_1^*}, \quad (2.21)$$

where $\langle \cdot, \cdot \rangle_{s,\mathcal{D}}$, $s > 0$ represents the duality pairing between $H^s(\mathcal{D})$ and its dual space $(H^s(\mathcal{D}))^*$ as the space of anti-linear functionals in a domain $\mathcal{D}$ with $L^2(\mathcal{D})$ pivot space, i.e., $\langle \cdot, \cdot \rangle_{0,\mathcal{D}} = (\cdot, \cdot)_{\mathcal{D}}$. It then follows from Lemma 2.4 that

$$\begin{aligned} |\langle \frac{\partial}{\partial \nu_{\Omega_1^*}} \mathbb{N}_1(\phi), w\rangle_{1/2,\Gamma_1}| &\le C(\|\mathbb{N}_1(\phi)\|_{H^1(\Omega_1^*)} + \|\phi\|_{\widetilde{H}^{-1}(\Omega_1^*)})\|\tilde{w}\|_{H^1(\Omega_1^*)} \\ &\le \frac{C}{\mu_{\min}} \|\phi\|_{\widetilde{H}^{-1}(\Omega_1^*)} \|w\|_{H^{1/2}(\Gamma_1)}, \end{aligned} \quad (2.22)$$

which implies the continuity of the normal derivative of the Newton potential into $H^{-1/2}(\Gamma_1)$. Since $\mathbb{N}_1(\phi)$ does not involve the N-th mode, neither does the normal derivative of $\mathbb{N}_1(\phi)$ on Γ_1, which shows that $\partial \mathbb{N}_1(\phi)/\partial \nu_{\Omega_1^*}$ lies in $\widetilde{H}^{-1/2}_{\neq N}(\Gamma_1)$ and the proof is completed. □

The single and double layer potentials have the following mapping properties.

Lemma 2.6. *It holds that*

$$\mathbb{S}_j : H^{-1/2}(\Gamma_j) \to H^1_{\neq N}(\Omega_j^*),$$
$$\mathbb{D}_j : H^{1/2}(\Gamma_j) \to H^1_{\neq N}(\Omega_j^*)$$

are continuous,

$$\|\mathbb{S}_j(\psi)\|_{H^1(\Omega_j^*)} \le \frac{C}{\mu_{\min}} \|\psi\|_{H^{-1/2}(\Gamma_j)} \quad \text{for } \psi \in H^{-1/2}(\Gamma_j),$$
$$\|\mathbb{D}_j(\psi)\|_{H^1(\Omega_j^*)} \le \frac{C}{\mu_{\min}} \|\psi\|_{H^{1/2}(\Gamma_j)} \quad \text{for } \psi \in H^{1/2}(\Gamma_j).$$

If a near-cutoff mode does not exist in ψ, then $\mu_{\min}$ is not involved.

PROOF. For $\psi \in H^{-1/2}(\Gamma_j)$ and $\phi \in C_0^\infty(\Omega_j^*)$, we obtain by using Lemma 2.4 that

$$\begin{aligned} \langle \mathbb{S}_j(\psi), \phi\rangle_{1,\Omega_j^*} &= \iint_{\Omega_j^*} \left[\int_{\Gamma_j} G(x,y)\psi(y) dy_2 \right] \bar{\phi}(x) dx \\ &= \int_{\Gamma_j} \psi(y) \left[\iint_{\Omega_j^*} G(x,y)\bar{\phi}(x) dx \right] dy_2 \\ &\le \|\psi\|_{H^{-1/2}(\Gamma_j)} \|\mathbb{N}_j(\bar{\phi})\|_{H^{1/2}(\Gamma_j)} \le \frac{C}{\mu_{\min}} \|\psi\|_{H^{-1/2}(\Gamma_j)} \|\phi\|_{\widetilde{H}^{-1}(\Omega_j^*)}. \end{aligned}$$

Since $C_0^\infty(\Omega_j^*)$ is dense in $\widetilde{H}^{-1}(\Omega_j^*)$, it follows that

$$\|\mathbb{S}_j(\psi)\|_{H^1(\Omega_j^*)} = \sup_{0\neq\phi\in C_0^\infty(\Omega_j^*)} \frac{|\langle \mathbb{S}_j(\psi),\phi\rangle_{1,\Omega_j^*}|}{\|\phi\|_{\widetilde{H}^{-1}(\Omega_j^*)}} \leq \frac{C}{\mu_{\min}}\|\psi\|_{H^{-1/2}(\Gamma_j)},$$

which is the required estimate for the single layer potential. Finally, since $\mathbb{S}_j(\psi)$ does not have the N-th mode, it is in $H^1_{\neq N}(\Omega_j^*)$.

For the double layer potential, we take $\psi \in H^{1/2}(\Gamma_j)$ and $\phi \in C_0^\infty(\Omega_j^*)$. Then, noting that $\nu_{\Omega_{12}} = \nu_{\Omega_j^*}$ on Γ_j, it can be shown by Lemma 2.5 that

$$\begin{aligned}
\langle \mathbb{D}_j(\psi),\phi\rangle_{1,\Omega_j^*} &= \iint_{\Omega_j^*}\left[\int_{\Gamma_j}\frac{\partial G}{\partial\nu_{\Omega_{12},y_1}}(x,y)\psi(y)dy_2\right]\bar{\phi}(x)dx \\
&= \int_{\Gamma_j}\psi(y)\frac{\partial}{\partial\nu_{\Omega_j^*,y_1}}\left[\iint_{\Omega_j^*}G(x,y)\bar{\phi}(x)dx\right]dy_2 \\
&\leq \|\psi\|_{H^{1/2}(\Gamma_j)}\|\frac{\partial}{\partial\nu_{\Omega_j^*}}\mathbb{N}_j(\bar{\phi})\|_{H^{-1/2}(\Gamma_j)} \leq \frac{C}{\mu_{\min}}\|\psi\|_{H^{1/2}(\Gamma_j)}\|\phi\|_{\widetilde{H}^{-1}(\Omega_j^*)}.
\end{aligned}$$

Since $C_0^\infty(\Omega_j^*)$ is dense in $\widetilde{H}^{-1}(\Omega_j^*)$, it follows that

$$\|\mathbb{D}_j(\psi)\|_{H^1(\Omega_j^*)} = \sup_{0\neq\phi\in C_0^\infty(\Omega_j^*)} \frac{|\langle \mathbb{D}_j(\psi),\phi\rangle_{1,\Omega_j^*}|}{\|\phi\|_{\widetilde{H}^{-1}(\Omega_j^*)}} \leq \frac{C}{\mu_{\min}}\|\psi\|_{H^{1/2}(\Gamma_j)},$$

which completes the proof of the continuity of the the double layer potential. At last, as the N-th mode is not involved in $\mathbb{D}_j(\psi)$, $\mathbb{D}_j(\psi)$ is in $H^1_{\neq N}(\Omega_j^*)$ and the proof is completed. □

Now we can estimate the non-cutoff components of the right-going and the left-going components of the solution.

Lemma 2.7. *It holds that*

$$\|u^{\text{right}}_{\neq N}\|^2_{H^{1/2}(\Gamma_1)} + \|u^{\text{left}}_{\neq N}\|^2_{H^{1/2}(\Gamma_2)} \leq \frac{C}{\mu^2_{\min}}\|u^{ex}\|^2_{H^1(\Omega)}.$$

If a near-cutoff mode does not exist in u^{right} *and* u^{left}*, then* $\mu_{\min}$ *is not involved.*

PROOF. Let $\eta = \min\{1, L\}$ and $\Omega_{1,\eta} = (0,\eta)\times\Theta = \Omega_{12}\cap\Omega_1^*$. The formulas (2.15) says

$$u^{\text{right}}_{\neq N} = \mathbb{S}_1(\frac{\partial u^{ex}}{\partial\nu_{\Omega_{12}}}) - \mathbb{D}_1(u^{ex})$$

and Lemma 2.6 yields that

$$\begin{aligned}
\|\mathbb{S}_1(\frac{\partial u^{ex}}{\partial\nu_{\Omega_{12}}})\|_{H^1(\Omega_{1,\eta})} &\leq \|\mathbb{S}_1(\frac{\partial u^{ex}}{\partial\nu_{\Omega_{12}}})\|_{H^1(\Omega_1^*)} \leq \frac{C}{\mu_{\min}}\|\frac{\partial u^{ex}}{\partial\nu_{\Omega_{12}}}\|_{H^{-1/2}(\Gamma_1)}, \\
\|\mathbb{D}_1(u^{ex})\|_{H^1(\Omega_{1,\eta})} &\leq \|\mathbb{D}_1(u^{ex})\|_{H^1(\Omega_1^*)} \leq \frac{C}{\mu_{\min}}\|u^{ex}\|_{H^{1/2}(\Gamma_1)}.
\end{aligned}$$

It then follows that

$$\|u_{\neq N}^{\text{right}}\|_{H^1(\Omega_{1,\eta})} \le \frac{C}{\mu_{\min}}(\|\frac{\partial u^{ex}}{\partial \nu_{\Omega_{12}}}\|_{H^{-1/2}(\Gamma_1)} + \|u^{ex}\|_{H^{1/2}(\Gamma_1)}) \le \frac{C}{\mu_{\min}}\|u^{ex}\|_{H^1(\Omega)}. \tag{2.23}$$

Here we used the estimates $\|\frac{\partial u^{ex}}{\partial \nu_{\Omega_{12}}}\|_{H^{-1/2}(\Gamma_1)} \le C\|u^{ex}\|_{H^1(\Omega)}$ of the normal derivative of u^{ex} on Γ_1 resulting from

$$|\langle -\frac{\partial u^{ex}}{\partial \nu_{\Omega_{12}}}, \phi\rangle_{1/2,\Gamma_1}| = |(\nabla u^{ex}, \nabla \phi)_{\Omega_1} - k^2(u^{ex}, \phi)_{\Omega_1}| \le C\|u^{ex}\|_{H^1(\Omega_1)}\|\phi\|_{H^1(\Omega_1)}$$

for $\phi \in H^1(\Omega_1)$. As the same estimate for $u_{\neq N}^{\text{left}}$ holds true, the desired estimate follows. □

Remark 2.8. *Since the series representations of $u_{\neq N}^{\text{right}}$ and $u_{\neq N}^{\text{left}}$ given in Ω_{12} can be extended to the domains Ω_1^δ and Ω_2^δ, Lemma 2.7 for the extended $u_{\neq N}^{\text{right}}$ and $u_{\neq N}^{\text{left}}$ still holds with Γ_j replaced by Γ_j^δ.*

2.4. Estimates of cutoff components

We will estimate the cutoff components of u^{right} and u^{left},

$$u_N^{\text{right}}(x_1, \cdot) = u_N^{ex}|_{\Gamma_1}\mathbb{L}_0(x_1)Y_N \quad \text{and} \quad u_N^{\text{left}}(x_1, \cdot) = u_N^{ex}|_{\Gamma_2}\mathbb{L}_L(x_1)Y_N.$$

Lemma 2.9. *The cutoff components satisfy the estimates*

$$\|u_N^{\text{right}}\|_{H^{1/2}(\Gamma_1)} \le C\|u^{ex}\|_{H^1(\Omega)} \quad \textit{and} \quad \|u_N^{\text{left}}\|_{H^{1/2}(\Gamma_2)} \le C\|u^{ex}\|_{H^1(\Omega)}.$$

PROOF. Since $u_N^{\text{right}} = u_N^{ex}|_{\Gamma_1}$, we have

$$\|u_N^{\text{right}}\|_{H^{1/2}(\Gamma_1)} = \|u_N^{ex}\|_{H^{1/2}(\Gamma_1)} \le \|u^{ex}\|_{H^{1/2}(\Gamma_1)} \le C\|u^{ex}\|_{H^1(\Omega)},$$

For u_N^{left} we use the fact $u_N^{\text{left}} = u_N^{ex}|_{\Gamma_2}$ to derive the same estimate as above, which completes the proof. □

2.5. Stability

We invoke all estimates in the previous subsections to establish the stability result of the solution to the problem (2.9)-(2.11).

Theorem 2.10. *The unique solution $(u, u_1, u_2) \in W$ to the problem* (2.9)-(2.11) *satisfies*

$$\|(u, u_1, u_2)\|_W \le C\|f\|_{L^2(\Omega)}. \tag{2.24}$$

PROOF. We know that $u = u^{ex}|_\Omega$, $u_1 = u^{\text{right}}|_{\Gamma_1}$ and $u_2 = u^{\text{left}}|_{\Gamma_2}$. Due to the orthogonality of Y_n, it holds that

$$\|u^{\text{right}}\|^2_{H^{1/2}(\Gamma_1)} = \|u_N^{\text{right}}\|^2_{H^{1/2}(\Gamma_1)} + \|u_{\neq N}^{\text{right}}\|^2_{H^{1/2}(\Gamma_1)}.$$

By using Lemma 2.7, Lemma 2.9 and (2.12), we prove that

$$\|u^{\text{right}}\|_{H^{1/2}(\Gamma_1)}, \quad \|u^{\text{left}}\|_{H^{1/2}(\Gamma_2)} \le \frac{C}{\mu_{\min}}\|f\|_{L^2(\Omega)}.$$

Thus, combining it with the stability (2.12) leads us to the stability estimate (2.24). □

2.6. Variational reformulation

In this subsection we reformulate the problem (2.9)-(2.11) in a weak form. The problem (2.9)-(2.11) can be written as a variational problem that seeks for $(u, u_1, u_2) \in W$ satisfying

$$A_\gamma((u, u_1, u_2), (v, v_1, v_2)) = (f, v)_\Omega \quad \text{for all } (v, v_1, v_2) \in W, \tag{2.25}$$

where

$$A_\gamma((u, u_1, u_2), (v, v_1, v_2)) = a((u, u_1, u_2), v) - \gamma b((u, u_1, u_2), (v_1, v_2))$$

with

$$\begin{aligned} a((u, u_1, u_2), v) &= (\nabla u, \nabla v)_\Omega - k^2(u, v)_\Omega - \sum_{i,j=1,2} \langle T_{ij}(u_i), v\rangle_{1/2,\Gamma_j}, \\ b((u, u_1, u_2), (v_1, v_2)) &= \sum_{i,j=1,2,i\neq j} [u - u_j - P_{ij}(u_i), v_j]_{\Gamma_j} \end{aligned}$$

for a constant $\gamma \in \mathbb{R}$. Here $[\cdot, \cdot]_{\Gamma_j}$ stands for the $H^{1/2}(\Gamma_j)$-inner product defined by

$$[\phi, \psi]_{\Gamma_j} = \sum_{n=0}^{\infty} (1 + \lambda_n^2)^{1/2} \phi_n \bar{\psi}_n$$

for $\phi = \sum_{n=0}^{\infty} \phi_n Y_n$ and $\psi = \sum_{n=0}^{\infty} \psi_n Y_n$ in $H^{1/2}(\Gamma_j)$. We note that since the problem (2.25) splits into two parts independent of $\gamma \in \mathbb{R}$

$$\begin{aligned} &a((u, u_1, u_2), v) = (f, v)_\Omega \text{ for } v \in H^1(\Omega), \\ &b((u, u_1, u_2), (v_1, v_2)) = 0 \text{ for } (v_1, v_2) \in H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_2), \end{aligned}$$

the problem (2.25) for different constants γ are all equivalent to each other. Consequently, Theorem 2.3 and Theorem 2.10 establish the following result.

Lemma 2.11. *The problem* (2.25) *for any* $\gamma \in \mathbb{R}$ *has a unique solution* (u, u_1, u_2) *in* W *satisfying the stability estimate* (2.24)*, and solutions for different* $\gamma \in \mathbb{R}$ *all coincide with each other.*

2.7. Adjoint problem

The subsection is devoted to introducing a reduced problem for the adjoint problem, which is analogous to the problem (2.25). The reduced problem is obtained by removing the domain Ω_{12} from the adjoint problem

$$\mathcal{A}(\phi, u) = (\phi, f)_\Omega \text{ for } \phi \in H^1(\widetilde{\Omega}) \tag{2.26}$$

of (2.2). The result in this subsection will be used for the error analysis based on the duality argument in studying the approximation method in the next section.

Let $T_{ij}^* : H^{1/2}(\Gamma_j) \to H^{-1/2}(\Gamma_i)$ and $P_{ij}^* : H^{-s}(\Gamma_j) \to H^{-s}(\Gamma_i)$ be the adjoint operators of $T_{ij} : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_j)$ and $P_{ij} : H^s(\Gamma_i) \to H^s(\Gamma_j)$ for $-1/2 \le s \le 1/2$, respectively, that is, for $\phi_i \in H^{1/2}(\Gamma_i)$ and $\psi_j \in H^{1/2}(\Gamma_j)$

$$\langle T_{ij}(\phi_i), \psi_j\rangle_{1/2,\Gamma_j} = \langle \phi_i, T_{ij}^*(\psi_j)\rangle_{1/2,\Gamma_i},$$

and for $\phi_i \in H^s(\Gamma_i)$ and $\psi_j \in H^{-s}(\Gamma_j)$ with $-1/2 \le s \le 1/2$

$$\langle P_{ij}(\phi_i), \psi_j\rangle_{s,\Gamma_j} = \langle \phi_i, P_{ij}^*(\psi_j)\rangle_{s,\Gamma_i}.$$

These adjoint operators are, in fact, defined as

$$T_{ij}^*(\psi) = \frac{\psi_N}{L} Y_N + \sum_{n \ne N} -\overline{\mathrm{i}\mu_n} e^{\overline{\mathrm{i}\mu_n} L} \psi_n Y_n, \qquad T_{jj}^*(\psi) = -\frac{\psi_N}{L} Y_N + \sum_{n \ne N} \overline{\mathrm{i}\mu_n} \psi_n Y_n,$$

for $\psi \in H^{1/2}(\Gamma_i)$. Also, for $i \ne j$ and $-1/2 \le s \le 1/2$

$$P_{ij}^*(\phi) = \sum_{n \ne N} e^{\overline{\mathrm{i}\mu_n} L} \phi_n Y_n \quad \text{for } \phi = \sum_{n=0}^{\infty} \phi_n Y_n \in H^s(\Gamma_j).$$

We study a reduced problem posed on Ω for the problem (2.26) by using the adjoint operators T_{ij}^* and P_{ij}^* instead of T_{ij} and P_{ij}. To this end, we consider the decomposition of the solution in Ω_{12},

$$\begin{aligned} u^{ex}(x_1, \cdot) &= u_N^{ex}|_{\Gamma_2} \mathbb{L}_L(x_1) Y_N + \sum_{n \ne N} A_n^* e^{-\overline{\mathrm{i}\mu_n} x_1} Y_n \quad (:= u^{*\text{right}}) \\ &\quad + u_N^{ex}|_{\Gamma_1} \mathbb{L}_0(x_1) Y_N + \sum_{n \ne N} B_n^* e^{\overline{\mathrm{i}\mu_n} x_1} Y_n \quad (:= u^{*\text{left}}) \end{aligned}$$

for some constants A_n^* and B_n^*. Denoting $w_1 = u^{*\text{left}}|_{\Gamma_1}$ and $w_2 = u^{*\text{right}}|_{\Gamma_2}$, we can see

$$\frac{\partial u^{ex}}{\partial \nu_{\Omega_1}} = T_{11}^*(w_1) + T_{12}^*(w_2) \quad \text{on } \Gamma_1 \quad \text{and} \quad \frac{\partial u^{ex}}{\partial \nu_{\Omega_2}} = T_{21}^*(w_1) + T_{22}^*(w_2) \quad \text{on } \Gamma_2 \tag{2.27}$$

and

$$u^{ex} = w_1 + P_{12}^*(w_2) \quad \text{on } \Gamma_1 \quad \text{and} \quad u^{ex} = w_2 + P_{21}^*(w_1) \quad \text{on } \Gamma_2.$$

Therefore, the adjoint problem (2.26) can be reduced to a problem seeking for $(w, w_1, w_2) \in W$ satisfying

$$A_\gamma^*((v, v_1, v_2), (w, w_1, w_2)) = (v, f)_\Omega \text{ for } (v, v_1, v_2) \in W, \tag{2.28}$$

where

$$\begin{aligned} A_\gamma^*((v, v_1, v_2), (w, w_1, w_2)) &= (\nabla v, \nabla w)_\Omega - k^2 (v, w)_\Omega \\ &\quad - \sum_{i,j=1,2} \langle v, T_{ij}^*(w_j)\rangle_{\Gamma_i} - \gamma \sum_{i,j=1,2, i \ne j} [v_i, w - w_i - P_{ij}^* w_j]_{\Gamma_i} \end{aligned}$$

for a constant $\gamma \in \mathbb{R}$.

Lemma 2.12. *For any $\gamma \in \mathbb{R}$, there exists a unique solution $(w, w_1, w_2) \in W$ independent of γ to the problem (2.28) satisfying*

$$\|(w, w_1, w_2)\|_W \leq C\|f\|_{L^2(\Omega)}.$$

In addition, for $g \in L^2(\Gamma_j)$ there exists a unique solution $(w, w_1, w_2) \in W$ to the problem

$$A^*_\gamma((v, v_1, v_2), (w, w_1, w_2)) = (v, g)_{\Gamma_j} \text{ for } (v, v_1, v_2) \in W \tag{2.29}$$

satisfying

$$\|(w, w_1, w_2)\|_W \leq C\|g\|_{L^2(\Gamma_j)}.$$

PROOF. The same arguments as those used for the primal problem (2.25) in the proceeding subsections provide the well-posedness of the problem (2.28) independent of $\gamma \in \mathbb{R}$.

On the other hand, we note that the problem (2.29) is a variational formulation of the Helmholtz equation in Ω with $f = 0$ but a source g is provided on the boundary Γ_j in the form

$$\frac{\partial w}{\partial \nu_{\Omega_j}} = T^*_{jj}(w_j) + T^*_{ji}(w_i) + g \quad \text{on } \Gamma_j$$

with the other boundary condition of (2.27) on Γ_i ($i \neq j$) unchanged. This problem is, in turn, equivalent to the full problem (2.26) with a line source $g\delta_{\Gamma_j}$ on Γ_j. Since the line source is in $\widetilde{H}^{-1}(\widetilde{\Omega})$ and the problem (2.26) is well-posed for $f = g\delta_{\Gamma_j} \in \widetilde{H}^{-1}(\widetilde{\Omega})$, the assertion can follow. □

3. Approximate method and convergence

The boundary conditions of the problem (2.9)-(2.11) resulting from truncating Ω_{12} involve the DtN and DtD operators, T_{ij} and P_{ij}, defined by infinite series. In order to apply a discretization technique such as the finite element method, the DtN and DtD operators need to be approximated by truncated series. We define approximate operators for the DtN and DtD operators,

$$T^M_{ij}(\psi) = \frac{\psi_N}{L} + \sum_{n=0, n\neq N}^{M} -\mathrm{i}\mu_n e^{\mathrm{i}\mu_n L}\psi_n Y_n, \qquad T^M_{ii}(\psi) = -\frac{\psi_N}{L} + \sum_{n=0, n\neq N}^{M} \mathrm{i}\mu_n \psi_n Y_n$$

for $\psi \in H^{1/2}(\Gamma_i)$. Also, for $i \neq j$ and $-1/2 \leq s \leq 1/2$

$$P^M_{ij}(\phi) = \sum_{n=0, n\neq N}^{M} e^{\mathrm{i}\mu_n L}\phi_n Y_n \quad \text{for } \phi = \sum_{n=0}^{\infty} \phi_n Y_n \in H^s(\Gamma_i).$$

By replacing the exact boundary conditions (2.10) and (2.11) with the truncated counterparts we introduce a problem in a variational form for an approximate solution $(u^M, u^M_1, u^M_2) \in W$ satisfying

$$A^M_\gamma((u^M, u^M_1, u^M_2), (v, v_1, v_2)) = (f, v)_\Omega \quad \text{for all } (v, v_1, v_2) \in W, \tag{3.1}$$

where

$$A_\gamma^M((u,u_1,u_2),(v,v_1,v_2)) = a^M((u,u_1,u_2),v) - \gamma b^M((u,u_1,u_2),(v_1,v_2))$$

with

$$a^M((u,u_1,u_2),v) = (\nabla u, \nabla v)_\Omega - k^2(u,v)_\Omega - \sum_{i,j=1,2} \langle T_{ij}^M(u_i), v\rangle_{1/2,\Gamma_j},$$

$$b^M((u,u_1,u_2),(v_1,v_2)) = \sum_{i,j=1,2,i\neq j} [u - u_j - P_{ij}^M(u_i), v_j]_{\Gamma_j}.$$

As done for the problem (2.25) in Subsection 2.6, the problem (3.1) also can split into two parts for any $\gamma \in \mathbb{R}$,

$$a^M((u,u_1,u_2),v) = (f,v)_\Omega \text{ for } v \in H^1(\Omega), \tag{3.2}$$

$$b^M((u,u_1,u_2),(v_1,v_2)) = 0 \text{ for } (v_1,v_2) \in H^{1/2}(\Gamma_1) \times H^{1/2}(\Gamma_2). \tag{3.3}$$

It implies that the problem (3.1) for different constants γ are all equivalent to each other, and allows us to infer that if the problem with a certain constant γ has a unique solution in W, then all other problems (3.1) with any γ share the same unique solution. Thus, we will establish that there exists a constant $\gamma \in \mathbb{R}$ such that the problem (3.1) is well-posed and solutions to the problem (3.1) converge exponentially to the solution to the problem (2.25) as M tends toward infinity.

Remark 3.1. *The boundary condition* (3.3) *can be imposed by using the $L^2(\Gamma_j)$-inner product instead of the $H^{1/2}(\Gamma_j)$-inner product. It results in an equivalent problem to find $(u,u_1,u_2) \in W$ satisfying*

$$\widetilde{A}^M((u,u_1,u_2),(v,v_1,v_2)) = (f,v)_\Omega \quad \text{for all } (v,v_1,v_2) \in V, \tag{3.4}$$

where $V = H^1(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_2)$ and $\widetilde{A}^M(\cdot,\cdot)$ is a sesquilinear form on $W \times V$ given by replacing $-\gamma b^M(\cdot,\cdot)$ in $A_\gamma^M(\cdot,\cdot)$ with $\tilde{b}^M(\cdot,\cdot)$ defined by

$$\tilde{b}^M((u,u_1,u_2),(v_1,v_2)) = (u - u_1 - P_{21}^M(u_2), v_1)_{\Gamma_1} + (u - P_{12}^M(u_1) - u_2, v_2)_{\Gamma_2}.$$

Since it is more appropriate in the well-posedness analysis to manipulate $H^{1/2}(\Gamma_j)$-norms of u_j than $L^2(\Gamma_j)$-norms, we take this problem (3.1) *in the analysis, however a finite element approximation will be applied to the problem* (3.4) *based on the L^2-inner product.*

3.1. Properties of the DtN and DtD operators and convergence of truncated operators

In this subsection, we examine some properties of the DtN and DtD operators and the exponential convergence of truncated operators. The first one is the commuting properties of T_{ij} and P_{ij}.

Lemma 3.2. *Let $i, j = 1, 2$ and $i \neq j$. It holds that*

$$T_{ii}P_{ji} = P_{ji}T_{jj} \tag{3.5}$$

and

$$T_{ij}P_{ji} = P_{ij}T_{ji}. \tag{3.6}$$

Here $T_{ij} : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_j)$ and P_{ij} is understood as an operator from $H^{1/2}(\Gamma_i)$ to $H^{1/2}(\Gamma_j)$ or from $H^{-1/2}(\Gamma_i)$ to $H^{-1/2}(\Gamma_j)$ depending on its position in the composition.

PROOF. For $\phi = \sum_{n=0}^{\infty} \phi_n Y_n$ in $H^{1/2}(\Gamma_j)$, it is easy to show that

$$T_{ii}P_{ji}(\phi) = \sum_{n=0}^{\infty} i\mu_n e^{\mathrm{i}\mu_n L}\phi_n Y_n = P_{ji}T_{jj}(\phi),$$

which leads to (3.5). Similarly, the second identity (3.6) can be obtained since

$$T_{ij}P_{ji}(\phi) = \sum_{n=0}^{\infty} -\mathrm{i}\mu_n e^{2\mathrm{i}\mu_n L}\phi_n Y_n = P_{ij}T_{ji}(\phi),$$

which completes the proof. □

For $u, w \in H^1(\Omega)$ and $u_i, w_i \in H^{1/2}(\Gamma_i)$, we denote the terms involving DtN operators in $A_\gamma(\cdot,\cdot)$ and $A_\gamma^*(\cdot,\cdot)$ by

$$\begin{aligned} \mathcal{T}((u_1, u_2), w) &= \sum_{i,j=1,2} \langle T_{ij}(u_i), w\rangle_{1/2,\Gamma_j}, \\ \mathcal{T}^*(u, (w_1, w_2)) &= \sum_{i,j=1,2} \langle u, T_{ij}^*(w_j)\rangle_{1/2,\Gamma_i}. \end{aligned} \tag{3.7}$$

These two sesquilinear forms have the duality property.

Lemma 3.3. *Assume that $u = u_i + P_{ji}(u_j)$ on Γ_i and $w = w_i + P_{ij}^*(w_j)$ on Γ_i for $u_i, w_i \in H^{1/2}(\Gamma_i)$, $i, j = 1, 2$, $i \neq j$. Then it holds that*

$$\mathcal{T}((u_1, u_2), w) = \mathcal{T}^*(u, (w_1, w_2)). \tag{3.8}$$

PROOF. It can be shown by Lemma 3.2 that for $i \neq j$,

$$\begin{aligned} \langle u, T_{ij}^*(w_j)\rangle_{1/2,\Gamma_i} &= \langle u_i + P_{ji}(u_j), T_{ij}^*(w_j)\rangle_{1/2,\Gamma_i} \\ &= \langle T_{ij}(u_i) + T_{ij}P_{ji}(u_j), w_j\rangle_{1/2,\Gamma_j} \\ &= \langle T_{ij}(u_i) + P_{ij}T_{ji}(u_j), w_j\rangle_{1/2,\Gamma_j} \\ &= \langle T_{ij}(u_i), w_j\rangle_{1/2,\Gamma_j} + \langle T_{ji}(u_j), P_{ij}^*(w_j)\rangle_{1/2,\Gamma_i}, \end{aligned}$$

from which it follows that

$$\begin{aligned}\sum_{i\neq j}\langle u, T_{ij}^*(w_j)\rangle_{1/2,\Gamma_i} &= \langle T_{ij}(u_i), w_j\rangle_{1/2,\Gamma_j} + \langle T_{ji}(u_j), P_{ij}^*(w_j)\rangle_{1/2,\Gamma_i} \\ &\quad + \langle T_{ji}(u_j), w_i\rangle_{1/2,\Gamma_i} + \langle T_{ij}(u_i), P_{ji}^*(w_i)\rangle_{1/2,\Gamma_j} \\ &= \sum_{i\neq j}\langle T_{ij}(u_i), w\rangle_{1/2,\Gamma_j}.\end{aligned} \tag{3.9}$$

By the similar way, we can show

$$\langle u, T_{ii}^*(w_i)\rangle_{1/2,\Gamma_i} = \langle T_{ii}(u_i), w_i\rangle_{1/2,\Gamma_i} + \langle T_{jj}(u_j), P_{ji}^*(w_i)\rangle_{1/2,\Gamma_j},$$

which yields

$$\sum_{i=1,2}\langle u, T_{ii}^*(w_i)\rangle_{1/2,\Gamma_i} = \sum_{i=1,2}\langle T_{ii}(u_i), w\rangle_{1/2,\Gamma_i}. \tag{3.10}$$

The proof is completed by combining (3.9) and (3.10). □

The truncated adjoint operators T_{ij}^{*M} and P_{ij}^{*M} can be defined analogously to T_{ij}^M and P_{ij}^M. Denoting by $\mathcal{T}^M$ and $\mathcal{T}^{*M}$ sesquilinear forms of (3.7) with T_{ij}, T_{ij}^* replaced by T_{ij}^M, T_{ij}^{*M}, respectively, one can easily show that the duality property for the truncated operators $\mathcal{T}^M$ and $\mathcal{T}^{*M}$ similar to (3.8) still holds as the truncated operators satisfy the same commuting properties as (3.5) and (3.6).

Lemma 3.4. *Assume that $u = u_i + P_{ji}^M(u_j)$ on Γ_i and $w = w_i + P_{ij}^{*M}(w_j)$ on Γ_i for $u_i, w_i \in H^{1/2}(\Gamma_i)$, $i, j = 1, 2$, $i \neq j$. Then it holds that*

$$\mathcal{T}^M((u_1, u_2), w) = \mathcal{T}^{*M}(u, (w_1, w_2)).$$

We will establish the exponential convergence of approximate DtN and DtD operators.

Lemma 3.5. *Let w be a radiating function in $H^1(\Omega_j^\delta)$ going out of Ω_j such that the non-cutoff component of w can be written as*

$$w_{\neq N}(x) = \sum_{n\neq N} w_n e^{(-1)^{j+1}\mathrm{i}\mu_n(x_1-\alpha)} Y_n(x_2) \quad \text{in } \Omega_j^\delta,$$

where $\alpha = -\delta$ for $j = 1$ and $\alpha = L + \delta$ for $j = 2$. If ϕ is a trace of w on Γ_j, then

$$\|(T_{ji} - T_{ji}^M)(\phi)\|_{H^{-1/2}(\Gamma_i)} \leq Ce^{-\tilde{\mu}_{M+1}\delta}\|w_{>M}\|_{H^{1/2}(\Gamma_j^\delta)}, \tag{3.11}$$

$$\|(P_{ji} - P_{ji}^M)(\phi)\|_{H^{1/2}(\Gamma_j)} \leq Ce^{-\tilde{\mu}_{M+1}\delta}\|w_{>M}\|_{H^{1/2}(\Gamma_j^\delta)} \tag{3.12}$$

for $M > N$.

PROOF. Since the n-th coefficients ϕ_n and w_n of ϕ and $w|_{\Gamma_j^\delta}$, respectively, satisfy

$$\phi_n = e^{-\tilde{\mu}_n \delta} w_n \text{ for } n > N,$$

it can be shown by using (2.8) that

$$\begin{aligned} \|(T_{ji} - T_{ji}^M)(\phi)\|^2_{H^{-1/2}(\Gamma_i)} &\le \sum_{n=M+1}^{\infty} (1+\lambda_n^2)^{-1/2} |\mu_n|^2 |\phi_n|^2 \\ &\le Ce^{-2\tilde{\mu}_{M+1}\delta} \sum_{n=M+1}^{\infty} (1+\lambda_n^2)^{1/2} |w_n|^2 \le Ce^{-2\tilde{\mu}_{M+1}\delta} \|w_{>M}\|^2_{H^{1/2}(\Gamma_j^\delta)}, \end{aligned} \tag{3.13}$$

which proves (3.11). In the first inequality, we used that $e^{-2\tilde{\mu}_{M+1}L} < 1$ in case that $i \ne j$. The convergence (3.12) of the DtD operators can be proved in the same way. □

The convergence of the truncated adjoint DtN operators can be derived in the same idea as the above.

Lemma 3.6. *Let w be the function in $H^1(\Omega_j^\delta)$ coming into Ω_j such that the non-cutoff component of w can be written as*

$$w_{\ne N}(x) = \sum_{n \ne N} w_n e^{(-1)^{j+1} \overline{\mathrm{i}\mu_n}(x_1 - \alpha)} Y_n(x_2) \quad \text{in } \Omega_j^\delta,$$

where $\alpha = -\delta$ for $j = 1$ and $\alpha = L + \delta$ for $j = 2$. If ϕ is a trace of w on Γ_j, then

$$\|(T_{ij}^* - T_{ij}^{*M})(\phi)\|_{H^{-1/2}(\Gamma_i)} \le Ce^{-\tilde{\mu}_{M+1}\delta} \|w_{>M}\|_{H^{1/2}(\Gamma_j^\delta)} \tag{3.14}$$

for $M > N$.

As a consequence of Lemma 3.5 and Remark 2.8 without a near-cutoff mode, we can have that for $j = 1, 2$

$$\begin{aligned} \|(T_{1j} - T_{1j}^M)(u^{\text{right}})\|_{H^{-1/2}(\Gamma_j)} &\le Ce^{-\tilde{\mu}_{M+1}\delta} \|u_{>M}^{\text{right}}\|_{H^{1/2}(\Gamma_1^\delta)} \le Ce^{-\tilde{\mu}_{M+1}\delta} \|u^{ex}\|_{H^1(\Omega)}, \\ \|(T_{2j} - T_{2j}^M)(u^{\text{left}})\|_{H^{-1/2}(\Gamma_j)} &\le Ce^{-\tilde{\mu}_{M+1}\delta} \|u_{>M}^{\text{left}}\|_{H^{1/2}(\Gamma_2^\delta)} \le Ce^{-\tilde{\mu}_{M+1}\delta} \|u^{ex}\|_{H^1(\Omega)}. \end{aligned} \tag{3.15}$$

The convergence of the Dirichlet data can also follow

$$\begin{aligned} \|(P_{12} - P_{12}^M)(u^{\text{right}})\|_{H^{-1/2}(\Gamma_2)} &\le Ce^{-\tilde{\mu}_{M+1}\delta} \|u^{ex}\|_{H^1(\Omega)}, \\ \|(P_{21} - P_{21}^M)(u^{\text{left}})\|_{H^{-1/2}(\Gamma_1)} &\le Ce^{-\tilde{\mu}_{M+1}\delta} \|u^{ex}\|_{H^1(\Omega)}. \end{aligned} \tag{3.16}$$

Similarly, by Lemma 3.6 it holds that for $j = 1, 2$

$$\begin{aligned} \|(T_{j1}^* - T_{j1}^{*M})(u^{*\text{left}})\|_{H^{-1/2}(\Gamma_j)} &\le Ce^{-\tilde{\mu}_{M+1}\delta} \|u_{>M}^{*\text{left}}\|_{H^{1/2}(\Gamma_1^\delta)} \le Ce^{-\tilde{\mu}_{M+1}\delta} \|u^{ex}\|_{H^1(\Omega)}, \\ \|(T_{j2}^* - T_{j2}^{*M})(u^{*\text{right}})\|_{H^{-1/2}(\Gamma_j)} &\le Ce^{-\tilde{\mu}_{M+1}\delta} \|u_{>M}^{*\text{right}}\|_{H^{1/2}(\Gamma_2^\delta)} \le Ce^{-\tilde{\mu}_{M+1}\delta} \|u^{ex}\|_{H^1(\Omega)}. \end{aligned} \tag{3.17}$$

since $u^{*\text{left}}_{>M} = u^{\text{right}}_{>M}$ and $u^{*\text{right}}_{>M} = u^{\text{left}}_{>M}$ for $M > N$. These exponential convergence (3.15), (3.16) and (3.17) of the DtN and DtD operators for radiating functions will play a crucial role in the convergence analysis.

The next lemma deals with some properties of the DtD operators.

Lemma 3.7. *Assume that $i \neq j$ and $M > N$. Then it holds that*

$$\|(P_{ij} - P^M_{ij})(\phi)\|_{H^{1/2}(\Gamma_j)} \leq Ce^{-\tilde{\mu}_{M+1}L}\|\phi\|_{H^{1/2}(\Gamma_i)} \tag{3.18}$$

for $\phi \in H^{1/2}(\Gamma_i)$. In addition, for $0 < \epsilon < 1/2$ and $v_i \in H^{1/2}(\Gamma_i)$, it holds that

$$|[P^M_{ij}(v_i), v_j]_{\Gamma_j}| \leq C_p\|v_i\|_{H^{1/2-\epsilon}(\Gamma_i)}\|v_j\|_{H^{1/2-\epsilon}(\Gamma_j)} \tag{3.19}$$

for a positive constant C_p independent of M.

PROOF. The convergence (3.18) of P^M_{ij} for $\phi \in H^{1/2}(\Gamma_i)$ can be proved as those in Lemma 3.5. For (3.19), let $v_\ell = \sum_{n=0}^{\infty} v_{\ell,n}Y_n$, $\ell = 1, 2$. We note that there exists a constant C_p (that may depend on L but is independent of M) such that

$$\begin{array}{ll} (1+\lambda_n^2)^{1/2} \leq (1+k^2)^{\epsilon}(1+\lambda_n^2)^{1/2-\epsilon} \leq C_p(1+\lambda_n^2)^{1/2-\epsilon} & \text{for } n \leq N, \\ e^{-\tilde{\mu}_n L} \leq C_p(1+\lambda_n^2)^{-\epsilon} & \text{for } n > N. \end{array}$$

By using the inequalities, we derive that

$$\begin{aligned} |[P^M_{ij}(v_i), v_j]_{\Gamma_j}| &\leq \sum_{n=0}^{N}(1+\lambda_n^2)^{1/2}|v_{i,n}||v_{j,n}| + \sum_{n=N+1}^{M} e^{-\tilde{\mu}_n L}(1+\lambda_n^2)^{1/2}|v_{i,n}||v_{j,n}| \\ &\leq C_p\|v_i\|_{H^{1/2-\epsilon}(\Gamma_i)}\|v_j\|_{H^{1/2-\epsilon}(\Gamma_j)}, \end{aligned}$$

which completes the proof of (3.19). □

3.2. Existence and uniqueness of approximate solutions

In this subsection we will show that the problem (3.1) admits a unique solution in W for any constant $\gamma \in \mathbb{R}$ when M is sufficiently large. We first consider the uniqueness of solutions to the problem (3.1), which is established by following the idea as that used in [11].

Lemma 3.8. *For any constant $\gamma \in \mathbb{R}$, there exists a positive constant $M_{uniq} > N$ such that if $M > M_{uniq}$, then the problem (3.1) has at most one solution.*

PROOF. To prove the lemma, we use a proof by contradiction. As addressed in the introductory part of this section, it is enough to find a constant $\gamma \in \mathbb{R}$ and $M_{uniq} > N$ for which the solution to the problem (3.1) is unique for all $M > M_{uniq}$.

Suppose that there exists an increasing sequence of integers $M_n > N$ such that $M_n \to \infty$ as $n \to \infty$ and the problem (3.1) with $f = 0$ has a non-trivial solution $(u^{M_n}, u_1^{M_n}, u_2^{M_n})$ in W satisfying $\|(u^{M_n}, u_1^{M_n}, u_2^{M_n})\|_W = 1$ for each M_n. Since W is compactly embedded in $W^\epsilon := H^{1-\epsilon}(\Omega) \times H^{1/2-\epsilon}(\Gamma_1) \times H^{1/2-\epsilon}(\Gamma_2)$

for $0 < \epsilon < 1/2$, there exists a subsequence $(u^{M_n}, u_1^{M_n}, u_2^{M_n})$ (with the same notation for the sake of simple presentation) that converges to (u, u_1, u_2) in W^ϵ.

Step I: We will show that $(u, u_1, u_2) = 0$. To do this, recalling the definition of $\mathcal{T}$ and $\mathcal{T}^{M_n}$, we denote

$$\begin{aligned}
&\mathcal{D}(u,v) = (\nabla u, \nabla v)_\Omega - k^2(u,v)_\Omega,\\
&\mathcal{B}(u,(v_1,v_2)) = [u,v_1]_{\Gamma_1} + [u,v_2]_{\Gamma_2},\\
&\mathcal{P}((u_1,u_2),(v_1,v_2)) = [u_1 + P_{21}(u_2), v_1]_{\Gamma_1} + [P_{12}(u_1) + u_2, v_2]_{\Gamma_2},\\
&\mathcal{P}^{M_n}((u_1,u_2),(v_1,v_2)) = [u_1 + P_{21}^{M_n}(u_2), v_1]_{\Gamma_1} + [P_{12}^{M_n}(u_1) + u_2, v_2]_{\Gamma_2}.
\end{aligned}$$

Then it holds that

$$\begin{aligned}
&A_\gamma((u,u_1,u_2),(v,v_1,v_2))\\
&\qquad = \mathcal{D}(u,v) - \mathcal{T}((u_1,v_1),v) - \gamma\big[\mathcal{B}(u,(v_1,v_2)) - \mathcal{P}((u_1,u_2),(v_1,v_2))\big],\\
&A_\gamma^{M_n}((u,u_1,u_2),(v,v_1,v_2))\\
&\qquad = \mathcal{D}(u,v) - \mathcal{T}^{M_n}((u_1,v_1),v) - \gamma\big[\mathcal{B}(u,(v_1,v_2)) - \mathcal{P}^{M_n}((u_1,u_2),(v_1,v_2))\big].
\end{aligned}$$

From the equation

$$A_\gamma^{M_n}((u^{M_n}, u_1^{M_n}, u_2^{M_n}),(\phi,\phi_1,\phi_2)) = 0 \quad \text{for all } \phi \in C^\infty(\bar\Omega),\ \phi_i \in C^\infty(\bar\Gamma_i)$$

it can be shown that

$$\begin{aligned}
&\mathcal{D}(u^{M_n} - u, \phi) - \gamma\mathcal{B}(u^{M_n} - u, (\phi_1,\phi_2))\\
&-\mathcal{T}^{M_n}((u_1^{M_n} - u_1, u_2^{M_n} - u_2),\phi) - (\mathcal{T}^{M_n} - \mathcal{T})((u_1,u_2),\phi)\\
&+\gamma\mathcal{P}^{M_n}((u_1^{M_n} - u_1, u_2^{M_n} - u_2),(\phi_1,\phi_2)) + \gamma(\mathcal{P}^{M_n} - \mathcal{P})((u_1,u_2),(\phi_1,\phi_2))\\
&+A_\gamma((u,u_1,u_2),(\phi,\phi_1,\phi_2)) = 0.
\end{aligned} \tag{3.20}$$

We shall make estimates of all terms except the last one in (3.20), which reveals that they converge to zero as $n \to \infty$. We first use a generalized Schwarz inequality to show

$$|\mathcal{D}(u^{M_n} - u, \phi)| \le C\|u^{M_n} - u\|_{H^{1-\epsilon}(\Omega)}\|\phi\|_{H^{1+\epsilon}(\Omega)}. \tag{3.21}$$

Similarly, a generalized Schwarz inequality and a trace inequality give

$$\begin{aligned}
|\mathcal{B}(u^{M_n} - u, (\phi_1,\phi_2))| &\le \sum_{j=1,2} \|u^{M_n} - u\|_{H^{1/2-\epsilon}(\Gamma_j)}\|\phi_j\|_{H^{1/2+\epsilon}(\Gamma_j)}\\
&\le C\sum_{j=1,2} \|u^{M_n} - u\|_{H^{1-\epsilon}(\Omega)}\|\phi_j\|_{H^{1/2+\epsilon}(\Gamma_j)}.
\end{aligned} \tag{3.22}$$

The boundedness of duality pairings, the continuity of the operator $T_{ij}^{M_n}$ and a trace inequality come together to prove that

$$\begin{aligned}
|\mathcal{T}^{M_n}((u_1^{M_n} - u_1, u_2^{M_n} - u_2),\phi)| &\le \sum_{i,j=1,2} \|T_{ij}^{M_n}(u_i^{M_n} - u_i)\|_{H^{-1/2-\epsilon}(\Gamma_j)}\|\phi\|_{H^{1/2+\epsilon}(\Gamma_j)}\\
&\le C\sum_{j=1,2} \|u_j^{M_n} - u_j\|_{H^{1/2-\epsilon}(\Gamma_j)}\|\phi\|_{H^{1+\epsilon}(\Omega)}
\end{aligned} \tag{3.23}$$

and

$$
\begin{aligned}
|(\mathcal{T}^{M_n}-\mathcal{T})((u_1,u_2),\phi)| &\le C\sum_{i,j=1,2}\|(T_{ij}-T_{ij}^{M_n})(u_i)\|_{H^{-1/2-\epsilon}(\Gamma_j)}\|\phi\|_{H^{1/2+\epsilon}(\Gamma_j)}\\
&\le C\sum_{i,j=1,2}\frac{1}{(1+\lambda_{M_n+1}^2)^{\epsilon/2}}\|u_i\|_{H^{1/2}(\Gamma_i)}\|\phi\|_{H^{1/2+\epsilon}(\Gamma_j)}.
\end{aligned}
\tag{3.24}
$$

A generalized Schwarz inequality again yields that

$$
|\mathcal{P}^{M_n}((u_1^{M_n}-u_1,u_2^{M_n}-u_2),(\phi_1,\phi_2))| \le C\sum_{i,j=1,2}\|u_j^{M_n}-u_j\|_{H^{1/2-\epsilon}(\Gamma_j)}\|\phi_i\|_{H^{1/2+\epsilon}(\Gamma_i)},
\tag{3.25}
$$

and by (3.18) we are led to

$$
\begin{aligned}
|(\mathcal{P}^{M_n}-\mathcal{P})((u_1,u_2),(\phi_1,\phi_2))| &\le \sum_{i\neq j}\|(P_{ij}-P_{ij}^{M_n})(u_i)\|_{H^{1/2}(\Gamma_j)}\|\phi_j\|_{H^{1/2}(\Gamma_j)}\\
&\le \sum_{i\neq j} e^{-\tilde{\mu}_{M_n+1}L}\|u_i\|_{H^{1/2}(\Gamma_i)}\|\phi_j\|_{H^{1/2}(\Gamma_j)}.
\end{aligned}
\tag{3.26}
$$

Since all terms in the right hand sides of the above estimates (3.21)-(3.26) converge to 0 as $n\to\infty$, passing to the limit of (3.20) leads

$$
A_\gamma((u,u_1,u_2),(\phi,\phi_1,\phi_2))=0.
$$

The uniqueness of solutions to the problem (2.25) shows that $(u,u_1,u_2)=0$.
Step II: We will show that the convergence of $(u^{M_n},u_1^{M_n},u_2^{M_n})$ to 0 in W^ϵ contradicts that $\|(u^{M_n},u_1^{M_n},u_2^{M_n})\|_W=1$. We begin with the identity

$$
\begin{aligned}
0 &= A_\gamma^{M_n}((u^{M_n},u_1^{M_n},u_2^{M_n}),(u^{M_n},u_1^{M_n},u_2^{M_n}))\\
&= \|u^{M_n}\|_{H^1(\Omega)}^2-(k^2+1)\|u^{M_n}\|_{L^2(\Omega)}^2-\mathcal{T}^{M_n}((u_1^{M_n},u_2^{M_n}),u^{M_n})\\
&\quad -\gamma\Big([u^{M_n}-u_1^{M_n}-P_{21}^{M_n}(u_2^{M_n}),u_1^{M_n}]_{\Gamma_1}+[u^{M_n}-u_2^{M_n}-P_{12}^{M_n}(u_1^{M_n}),u_2^{M_n}]_{\Gamma_2}\Big).
\end{aligned}
\tag{3.27}
$$

By using the fact that $u^{M_n}=u_i^{M_n}+P_{ji}^{M_n}(u_j^{M_n})$ on Γ_i and invoking the definition of $T_{ij}^{M_n}$, we can show that for $u_1^{M_n}=\sum_{\ell=0}^\infty u_{1,\ell}^{M_n}Y_\ell$ and $u_2^{M_n}=\sum_{\ell=0}^\infty u_{2,\ell}^{M_n}Y_\ell$

$$
\begin{aligned}
\mathcal{T}^{M_n}((u_1^{M_n},u_2^{M_n}),u^{M_n}) &= \sum_{i,j=1,2}\langle T_{ij}^{M_n}(u_i^{M_n}),u^{M_n}\rangle_{1/2,\Gamma_j}\\
&= \sum_{\ell=0}^{N-1}4\mu_\ell\Im(e^{\mathrm{i}\mu_\ell L})\Re(u_{1,\ell}^{M_n}\bar{u}_{2,\ell}^{M_n})-\sum_{\ell=N+1}^{M_n}\tilde{\mu}_\ell(1-e^{-2\tilde{\mu}_\ell L})(|u_{1,\ell}^{M_n}|^2+|u_{2,\ell}^{M_n}|^2)\\
&\quad +\frac{1}{L}\big(2\Re(u_{1,N}^{M_n}\bar{u}_{2,N}^{M_n})-(|u_{1,N}^{M_n}|^2+|u_{2,N}^{M_n}|^2)\big),
\end{aligned}
\tag{3.28}
$$

which is real. Since $\tilde{\mu}_\ell(1-e^{-2\tilde{\mu}_\ell L}) > 0$ for $\ell > N$ and the last term for the N-th mode is non-positive, by using the estimate of the finite sum

$$\sum_{\ell=0}^{N-1} 4\mu_\ell \Im(e^{\mathrm{i}\mu_\ell L})\Re(u_{1,\ell}^{M_n}\bar{u}_{2,\ell}^{M_n}) \leq C_t\|u_1^{M_n}\|_{H^{1/2-\epsilon}(\Gamma_1)}\|u_2^{M_n}\|_{H^{1/2-\epsilon}(\Gamma_2)}$$

for a constant $C_t > 0$ independent of M_n, we can obtain that

$$\mathcal{T}^{M_n}((u_1^{M_n}, u_2^{M_n}), u^{M_n}) \leq C_t\|u_1^{M_n}\|_{H^{1/2-\epsilon}(\Gamma_1)}\|u_2^{M_n}\|_{H^{1/2-\epsilon}(\Gamma_2)}. \tag{3.29}$$

On the other hand, the terms of the $H^{1/2}$-inner product in (3.27) are estimated by using (3.19), inequalities of arithmetic-geometric means and a trace inequality with a trace constant C_{tr}, as

$$\begin{aligned}&-\gamma\Re\Big([u^{M_n} - u_j^{M_n} - P_{ij}^{M_n}(u_i^{M_n}), u_j^{M_n}]_{\Gamma_j}\Big)\\ &\geq \gamma\Big(\frac{1}{2}\|u_j^{M_n}\|^2_{H^{1/2}(\Gamma_j)} - \frac{C_{tr}}{2}\|u^{M_n}\|^2_{H^1(\Omega)} - \frac{C_p}{2}\big(\|u_i^{M_n}\|^2_{H^{1/2-\epsilon}(\Gamma_i)} + \|u_j^{M_n}\|^2_{H^{1/2-\epsilon}(\Gamma_j)}\big)\Big).\end{aligned} \tag{3.30}$$

We use (3.29) and (3.30) in the real part of (3.27) to obtain

$$\begin{aligned}&\|u^{M_n}\|^2_{H^1(\Omega)} + \frac{\gamma}{2}(\|u_1^{M_n}\|^2_{H^{1/2}(\Gamma_1)} + \|u_2^{M_n}\|^2_{H^{1/2}(\Gamma_2)})\\ &\leq (k^2+1)\|u^{M_n}\|^2_{L^2(\Omega)} + \gamma C_{tr}\|u^{M_n}\|^2_{H^1(\Omega)} + (\gamma C_p + C_t)(\|u_1^{M_n}\|^2_{H^{1/2-\epsilon}(\Gamma_1)} + \|u_2^{M_n}\|^2_{H^{1/2-\epsilon}(\Gamma_2)}).\end{aligned}$$

We take a positive constant γ so that $\gamma C_{tr} < 1/2$ and have

$$\begin{aligned}C\|(u^{M_n}, u_1^{M_n}, u_2^{M_n})\|^2_W &\leq \frac{1}{2}\|u^{M_n}\|^2_{H^1(\Omega)} + \frac{\gamma}{2}(\|u_1^{M_n}\|^2_{H^{1/2}(\Gamma_1)} + \|u_2^{M_n}\|^2_{H^{1/2}(\Gamma_2)})\\ &\leq (k^2+1)\|u^{M_n}\|^2_{L^2(\Omega)} + (\gamma C_p + C_t)(\|u_1^{M_n}\|^2_{H^{1/2-\epsilon}(\Gamma_1)} + \|u_2^{M_n}\|^2_{H^{1/2-\epsilon}(\Gamma_2)}).\end{aligned}$$

Since the right hand side tends to zero as n goes to infinity, it contradicts the fact that $\|(u^{M_n}, u_1^{M_n}, u_2^{M_n})\|_W = 1$ and the proof is completed. □

Lemma 3.9. *Let M_{uniq} be the constant defined in Lemma 3.8. If $M > M_{uniq}$, then the problem* (3.1) *for any constant γ has a unique solution.*

PROOF. It suffices to show that there exists a constant γ for which the problem (3.1) with $M > M_{uniq}$ has a unique solution.

As presented in Step II of Lemma 3.8, by (3.29) and (3.30) with γ satisfying $\gamma C_{tr} < 1/2$, we can show that

$$\begin{aligned}&|A_\gamma^M((u,u_1,u_2),(u,u_1,u_2))| \geq \frac{1}{2}\|u\|^2_{H^1(\Omega)} + \frac{\gamma}{2}\big(\|u_1\|^2_{H^{1/2}(\Gamma_1)} + \|u_2\|^2_{H^{1/2}(\Gamma_2)}\big)\\ &\quad - (k^2+1)\|u\|^2_{L^2(\Omega)} - (\gamma C_p + C_t)\big(\|u_1\|^2_{H^{1/2-\epsilon}(\Gamma_1)} + \|u_2\|^2_{H^{1/2-\epsilon}(\Gamma_2)}\big),\end{aligned} \tag{3.31}$$

which is a Gårding type inequality for $A_\gamma^M(\cdot,\cdot)$. It implies that there exists a positive constant C independent of $(u,u_1,u_2)\in W$ such that

$$\|(u,u_1,u_2)\|_W \le C\left(\sup_{(v,v_1,v_2)\in W}\frac{|A_\gamma^M((u,u_1,u_2),(v,v_1,v_2))|}{\|(v,v_1,v_2)\|_W}+\|(u,u_1,u_2)\|_{W^\epsilon}\right)$$

with the obvious product norm $\|\cdot\|_{W^\epsilon}$ of W^ϵ. Since the mapping $(u,u_1,u_2)\mapsto A_\gamma^M((u,u_1,u_2),\cdot)$ from W to its dual space W^* as the space of anti-linear functionals is continuous and injective (by Lemma 3.8), and W is compactly embedded to W^ϵ, Peetre-Tartar lemma (see e.g., [9]) gives the inf-sup condition

$$\|(u,u_1,u_2)\|_W \le C\sup_{(v,v_1,v_2)\in W}\frac{|A_\gamma^M((u,u_1,u_2),(v,v_1,v_2))|}{\|(v,v_1,v_2)\|_W}$$

for some constant C. Since $A_\gamma^M((u,u_1,u_2),(v,v_1,v_2)) = A_\gamma^M((\bar v,\bar v_1,\bar v_2),(\bar u,\bar u_1,\bar u_2))$, the inf-sup condition for the adjoint problem holds as well and so the proof is completed. □

3.3. Convergence and stability of approximate solutions

We are in a position to prove the main convergence result of approximate solutions.

Theorem 3.10. *Assume that each cavity Ω_j includes a straight waveguide Ω_j^δ of width δ with Γ_j on its boundary, which does not include any inclusion or wave source, as in Figure 1. Let (u,u_1,u_2) and (u^M,u_1^M,u_2^M) in W be the solutions to the problems* (2.25) *and* (3.1)*, respectively. Then there exists a constant $M_{conv} > M_{uniq}$ such that for $M > M_{conv}$*

$$\|(u,u_1,u_2)-(u^M,u_1^M,u_2^M)\|_W \le Ce^{-\tilde\mu_{M+1}\delta}\|u\|_{H^1(\Omega)}. \tag{3.32}$$

PROOF. Let $(e^M,e_1^M,e_2^M) = (u,u_1,u_2)-(u^M,u_1^M,u_2^M)$ be the error function. The proof for the error estimate consists of five steps. Step I through Step IV are devoted to the estimation of the error e^M in Ω and Step V completes the proof by providing the error estimates of e_j^M in $H^{1/2}(\Gamma_j)$.
Step I: In Step I, we prove that for $M > N$

$$\begin{aligned}&\|e^M\|^2_{H^1(\Omega)}-(k^2+1)\|e^M\|^2_{L^2(\Omega)}\\ &\le C\left(e^{-\tilde\mu_{M+1}\delta}\|u\|_{H^1(\Omega)}\|e^M\|_{H^1(\Omega)}+\sum_{n=0}^{N-1}4\mu_n\Im(e^{\mathrm{i}\mu_nL})\Re(e^M_{1,n}\bar e^M_{2,n})\right).\end{aligned}\tag{3.33}$$

To do this, we begin with the equations for the error function

$$\mathcal{D}(e^M,v)-\mathcal{T}^M((e_1^M,e_2^M),v) = (\mathcal{T}-\mathcal{T}^M)((u_1,u_2),v)\quad\text{for all } v\in H^1(\Omega), \tag{3.34}$$

$$e^M-e_i^M-P_{ji}^M(e_j^M) = (P_{ji}-P_{ji}^M)(u_j)\quad\text{on }\Gamma_i,\quad i\ne j. \tag{3.35}$$

Taking $v = e^M$ gives

$$\mathcal{D}(e^M, e^M) = \mathcal{T}^M((e_1^M, e_2^M), e^M) + (\mathcal{T} - \mathcal{T}^M)((u_1, u_2), e^M). \tag{3.36}$$

For estimating the first term in the right-hand-side, we use a computation similar to that used for (3.28). The only difference comes from (3.35) having a non-zero right-hand-side, however since $T_{ij}^M(e_i^M)$ annihilates the range of $(P_{ij} - P_{ij}^M)$, $i \neq j$, the following required estimate can be obtained,

$$\mathcal{T}^M((e_1^M, e_2^M), e^M)) \leq \sum_{n=0}^{N-1} 4\mu_n \Im(e^{\mathrm{i}\mu_n L}) \Re(e_{1,n}^M \bar{e}_{2,n}^M). \tag{3.37}$$

For the second term of the right-hand-side in (3.36), noting that $u_1 = u^{\text{right}}$ on Γ_1, $u_2 = u^{\text{left}}$ on Γ_2 and $u = u^{ex}$ in Ω, we use (3.15) and a trace inequality to show that

$$|(\mathcal{T} - \mathcal{T}^M)((u_1, u_2), e^M)| \leq Ce^{-\tilde{\mu}_{M+1}\delta} \|u\|_{H^1(\Omega)} \|e^M\|_{H^1(\Omega)}. \tag{3.38}$$

The desired estimate (3.33) follows by taking the real part of (3.36) and combining (3.37)-(3.38).

Step II: In Step II, we prove that for $M > N$

$$\|e^M\|_{L^2(\Omega)} \leq Ce^{-\tilde{\mu}_{M+1}\delta} (\|u\|_{H^1(\Omega)} + \|e^M\|_{H^1(\Omega)}). \tag{3.39}$$

To do this, we will consider the adjoint problem with a source function e^M. Let $(w, w_1, w_2) \in W$ be the solution to the adjoint problem (2.28) with $f = e^M$, which also can be written as

$$\mathcal{D}(v, w) - \mathcal{T}^*(v, (w_1, w_2)) = (v, e^M)_\Omega \quad \text{for all } v \in H^1(\Omega), \tag{3.40}$$

$$w - w_i - P_{ij}^*(w_j) = 0 \quad \text{on } \Gamma_i, \quad i \neq j. \tag{3.41}$$

Lemma 2.12 gives the stability result

$$\|(w, w_1, w_2)\|_W \leq C\|e^M\|_{L^2(\Omega)}. \tag{3.42}$$

For $v = e^M$ in (3.40), we have

$$\|e^M\|_{L^2(\Omega)}^2 = \mathcal{D}(e^M, w) - \mathcal{T}^*(e^M, (w_1, w_2)). \tag{3.43}$$

On the other hand, by taking $v = w$ in (3.34) we are led to

$$\mathcal{D}(e^M, w) - \mathcal{T}^M((e_1^M, e_2^M), w) = (\mathcal{T} - \mathcal{T}^M)((u_1, u_2), w). \tag{3.44}$$

By using Lemma 3.4 with (3.35) and the fact that $T_{ij}^{*M}(w_j)$ annihilates the range of $P_{ij} - P_{ij}^M$, we can show that

$$\begin{aligned} \mathcal{T}^M((e_1^M, e_2^M), w) &= \mathcal{T}^{*M}(e^M, (w_1, w_2)) \\ &= \mathcal{T}^*(e^M, (w_1, w_2)) - (\mathcal{T}^* - \mathcal{T}^{*M})(e^M, (w_1, w_2)). \end{aligned}$$

It then follows from (3.43) and (3.44) that

$$\|e^M\|^2_{L^2(\Omega)} = (\mathcal{T} - \mathcal{T}^M)((u_1, u_2), w) - (\mathcal{T}^* - \mathcal{T}^{*M})(e^M, (w_1, w_2)). \tag{3.45}$$

As done for the estimate (3.38), we use (3.15), (3.17) and the stability (3.42) to estimate the terms in the right-hand-side,

$$\begin{aligned} |(\mathcal{T} - \mathcal{T}^M)((u_1, u_2), w)| &\le Ce^{-\tilde{\mu}_{M+1}\delta}\|u\|_{H^1(\Omega)}\|e^M\|_{L^2(\Omega)}, \\ |(\mathcal{T}^* - \mathcal{T}^{*M})(e^M, (w_1, w_2))| &\le Ce^{-\tilde{\mu}_{M+1}\delta}\|e^M\|_{H^1(\Omega)}\|e^M\|_{L^2(\Omega)}. \end{aligned} \tag{3.46}$$

Combining (3.45) and (3.46) completes the L^2 estimate (3.39) for e^M.
Step III: In Step III, we prove that for $M > N$

$$\sum_{n=0}^{N-1} 4\mu_n \Im(e^{\mathrm{i}\mu_n L})\Re(e^M_{1,n}\bar{e}^M_{2,n}) \le Ce^{-\tilde{\mu}_{M+1}\delta}(\|u\|_{H^1(\Omega)} + \|e^M\|_{H^1(\Omega)})\|e^M\|_{H^1(\Omega)}. \tag{3.47}$$

Denoting by π_j the projection onto the subspace spanned by $\{Y_n\}_{n=0}^{N-1}$ of $L^2(\Gamma_j)$, let $g_j = \pi_j(\partial e^M/\partial\nu_{\Omega_j}) \in L^2(\Gamma_j)$. As done in Step II, we consider the solution (w^j, w^j_1, w^j_2) in W to the adjoint problem (2.29) with $g = g_j$. The same arguments as those used in Step II lead to

$$\begin{aligned} |U| &:= |(e^M|_{\Gamma_1}, g_1)_{\Gamma_1} + (e^M|_{\Gamma_2}, g_2)_{\Gamma_2}| \\ &\le Ce^{-\tilde{\mu}_{M+1}\delta}(\|u\|_{H^1(\Omega)} + \|e^M\|_{H^1(\Omega)})(\|g_1\|_{L^2(\Gamma_1)} + \|g_2\|_{L^2(\Gamma_2)}). \end{aligned} \tag{3.48}$$

Here U is, in fact, the term we want to estimate in (3.47), in that,

$$\begin{aligned} U = -\mathrm{i}\mu_n \sum_{n=0}^{N-1} \Big[&(e^M_{1,n} + e^{\mathrm{i}\mu_n L}e^M_{2,n})(\bar{e}^M_{1,n} - e^{-\mathrm{i}\mu_n L}\bar{e}^M_{2,n}) \\ &+ (e^M_{2,n} + e^{\mathrm{i}\mu_n L}e^M_{1,n})(\bar{e}^M_{2,n} - e^{-\mathrm{i}\mu_n L}\bar{e}^M_{1,n})\Big] = \sum_{n=0}^{N-1} 4\mu_n \Im(e^{\mathrm{i}\mu_n L})\Re(e^M_{1,n}\bar{e}^M_{2,n}). \end{aligned}$$

Therefore using

$$\|g_j\|_{L^2(\Gamma_j)} \le C\|\frac{\partial e^M}{\partial \nu_{\Omega_j}}\|_{H^{-1/2}(\Gamma_j)} \le C\|e^M\|_{H^1(\Omega)},$$

we can obtain from (3.48) the desired estimate

$$\sum_{n=0}^{N-1} 4\mu_n \Im(e^{\mathrm{i}\mu_n L})\Re(e^M_{1,n}\bar{e}^M_{2,n}) \le Ce^{-\tilde{\mu}_{M+1}\delta}(\|u\|_{H^1(\Omega)} + \|e^M\|_{H^1(\Omega)})\|e^M\|_{H^1(\Omega)}.$$

Step IV: We prove that there exists a constant $M_{conv} > M_{uniq}$ such that for $M > M_{conv}$

$$\|e^M\|_{H^1(\Omega)} \le Ce^{-\tilde{\mu}_{M+1}\delta}\|u\|_{H^1(\Omega)}. \tag{3.49}$$

To prove it, we have only to combine all estimates (3.33), (3.39) and (3.47) obtained in the previous steps. Applying (3.39) and (3.47) to (3.33) we see that

$$
\begin{aligned}
&\|e^M\|_{H^1(\Omega)}^2 - C_1(k^2+1)e^{-2\tilde{\mu}_{M+1}\delta}(\|u\|_{H^1(\Omega)}^2 + \|e^M\|_{H^1(\Omega)}^2) \\
&\le C_2 e^{-\tilde{\mu}_{M+1}\delta}\|u\|_{H^1(\Omega)}\|e^M\|_{H^1(\Omega)} + C_3 e^{-\tilde{\mu}_{M+1}\delta}(\|u\|_{H^1(\Omega)} + \|e^M\|_{H^1(\Omega)})\|e^M\|_{H^1(\Omega)},
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
&\left(1 - \frac{\beta(C_2+C_3)}{2} - \left(C_1(k^2+1)e^{-2\tilde{\mu}_{M+1}\delta} + C_3 e^{-\tilde{\mu}_{M+1}\delta}\right)\right)\|e^M\|_{H^1(\Omega)}^2 \\
&\qquad \le \left(C_1(k^2+1) + \frac{(C_2+C_3)}{2\beta}\right)e^{-2\tilde{\mu}_{M+1}\delta}\|u\|_{H^1(\Omega)}^2
\end{aligned}
$$

with the help of the Cauchy-Schwarz inequality with any constant $\beta > 0$. Now, by taking a large M_{conv} and a small β such that

$$
\left(C_1(k^2+1)e^{-2\tilde{\mu}_{M+1}\delta} + C_3 e^{-\tilde{\mu}_{M+1}\delta}\right) < \frac{1}{4} \quad \text{for } M > M_{conv} \quad \text{and} \quad \frac{\beta(C_2+C_3)}{2} < \frac{1}{4},
$$

we can obtain the estimate (3.49) for e^M in Ω.

Step V: We prove that the errors e_j^M of auxiliary functions satisfy

$$
\|e_j^M\|_{H^{1/2}(\Gamma_j)} \le C\frac{e^{-\tilde{\mu}_{M+1}\delta}}{\mu_{\min}}\|u\|_{H^1(\Omega)} \tag{3.50}
$$

for $M > M_{conv}$.

We consider the decomposition of $e_j^M = e_{j,\le M}^M + e_{j,>M}^M$. Due to (3.35), the high frequency component $e_{j,>M}^M$ satisfies

$$
e_{j,>M}^M = e^M|_{\Gamma_j,>M} - (P_{ij} - P_{ij}^M)(u_i) \quad \text{on } \Gamma_j.
$$

By using a trace inequality, (3.16) and (3.49) it can be shown that

$$
\|e_{j,>M}^M\|_{H^{1/2}(\Gamma_j)} \le C\|e^M\|_{H^1(\Omega)} + Ce^{-\tilde{\mu}_{M+1}\delta}\|u\|_{H^1(\Omega)} \le Ce^{-\tilde{\mu}_{M+1}\delta}\|u\|_{H^1(\Omega)}. \tag{3.51}
$$

On the other hand, since the low frequency component $e_{j,\le M}^M$ satisfies

$$
\begin{aligned}
e^M|_{\Gamma_j,\le M} &= e_{j,\le M}^M + P_{ij}(e_{i,\le M}^M) \quad \text{on } \Gamma_j, \\
\frac{\partial e^M}{\partial \nu_{\Omega_j}}|_{\Gamma_j,\le M} &= T_{jj}(e_{j,\le M}^M) + T_{ij}(e_{i,\le M}^M) \quad \text{on } \Gamma_j,
\end{aligned}
$$

that is $e_{1,\le M}^M$ and $e_{2,\le M}^M$ are the Dirichlet traces on Γ_1 and Γ_2 of the right-going and left-going components of $e_{\le M}^M$ in Ω_{12}, respectively, the stability analysis in Lemma 2.7 based on the continuity of layer potentials for the non-cutoff component and Lemma 2.9 for the cutoff component still holds true and hence we are led to

$$
\|e_{j,\le M}^M\|_{H^{1/2}(\Gamma_j)} \le \frac{C}{\mu_{\min}}\|e_{\le M}^M\|_{H^1(\Omega_j^\delta)} \le \frac{C}{\mu_{\min}}\|e^M\|_{H^1(\Omega)} \le C\frac{e^{-\tilde{\mu}_{M+1}\delta}}{\mu_{\min}}\|u\|_{H^1(\Omega)} \tag{3.52}
$$

Finally, (3.51) and (3.52) yield that

$$\|e_j^M\|^2_{H^{1/2}(\Gamma_j)} = \|e_{j,\leq M}^M\|^2_{H^{1/2}(\Gamma_j)} + \|e_{j,>M}^M\|^2_{H^{1/2}(\Gamma_j)} \leq C\frac{e^{-2\tilde{\mu}_{M+1}\delta}}{\mu^2_{\min}}\|u\|^2_{H^1(\Omega)},$$

which completes the proof. □

Finally we can establish the stability of approximate solutions to the problem (3.1).

Theorem 3.11. *For any $M > M_{conv}$ the problem* (3.1) *has a unique solution (u^M, u_1^M, u_2^M) in W satisfying*

$$\|(u^M, u_1^M, u_2^M)\|_W \leq C\|f\|_{H^1(\Omega)}.$$

PROOF. By Lemma 3.9 and Theorem 3.10, there exists a unique solution (u^M, u_1^M, u_2^M) in W to the problem (3.1) satisfying the error estimate (3.32). The stability estimate in Theorem 2.10 for the exact solution $(u, u_1, u_2) \in W$ and a triangle inequality give

$$\|(u^M, u_1^M, u_2^M)\|_W \leq \|(e^M, e_1^M, e_2^M)\|_W + \|(u, u_1, u_2)\|_W \leq C\|f\|_{L^2(\Omega)},$$

which completes the proof. □

4. Numerical experiments

For numerical tests for convergence of approximate solutions, we consider the problem (with H a positive constant)

$$\begin{aligned} -\Delta u - k^2 u = 0 \quad & \text{in } \widetilde{\Omega} = (-H, H) \times (0,1), \\ \frac{\partial u}{\partial \nu_{\widetilde{\Omega}}} = 0 \quad & \text{on } (-H, H) \times \{0,1\} \end{aligned}$$

with Dirichlet conditions imposed on $\{\pm H\} \times (0,1)$ such that the exact solution is given by, if $k \neq n\pi$ for $n = 0, 1, \ldots,$

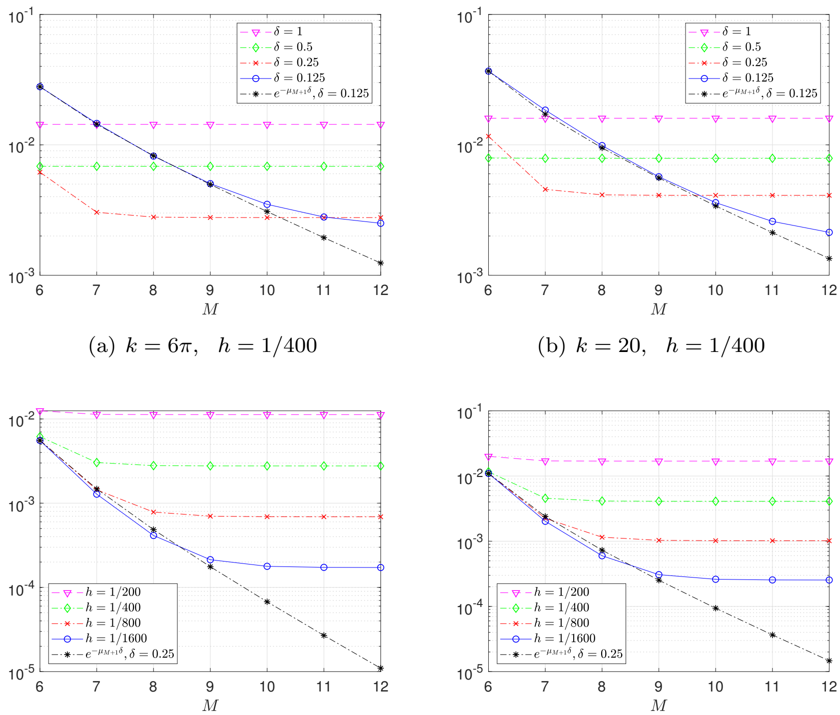
$$u^{ex}(x_1, x_2) = \sum_{n=0}^{49}(e^{\mathrm{i}\mu_n(x_1+H)} + 3e^{-\mathrm{i}\mu_n(x_2-H)})Y_n(x_2),$$

and if $k = N\pi$ for some $N < 50$

$$u^{ex}(x_1, x_2) = (2x_1 + 1)Y_N(x_2) + \sum_{n=0, n\neq N}^{49}(e^{\mathrm{i}\mu_n(x_1+H)} + 3e^{-\mathrm{i}\mu_n(x_2-H)})Y_n(x_2).$$

We conduct numerical experiments for

(1) Relative L^2-errors in Ω versus M,



(a) $k = 6\pi, \quad h = 1/400$

(b) $k = 20, \quad h = 1/400$

(c) $k = 6\pi, \quad \delta = 0.25$

(d) $k = 20, \quad \delta = 0.25$

Figure 2: Relative L^2-Errors vs. M

(2) Relative L^2-errors in Ω versus δ,
(3) Relative L^2-errors in Ω versus L,
(4) Relative L^2-errors in Ω versus $\mu_{\min}$

by applying the finite element method to the problem with the help of the finite element library `deal.II` [4]. As mentioned in Remark 3.1, for numerical computations we employ the variational problem associated with $\widetilde{A}^M(\cdot,\cdot)$,

$$\widetilde{A}^M((u,u_1,u_2),(v,v_1,v_2)) = a^M((u,u_1,u_2),v) + \tilde{b}^M((u,u_1,u_2),(v_1,v_2))$$

for (u,u_1,u_2) and $(v,v_1,v_2) \in W$, with $\gamma = -1$ and the boundary terms $\tilde{b}^M(\cdot,\cdot)$ with respect to the L^2-inner product rather than $A^M_\gamma(\cdot,\cdot)$ including $-\gamma b^M(\cdot,\cdot)$ with respect to the $H^{1/2}$-inner product.

Test (1): relative L^2-errors in Ω vs. M. In this test we set $k = 20$ for the case that cutoff modes are not involved and $k = 6\pi$ for the case that a cutoff mode exists. For the domain $\widetilde{\Omega}$ with $H = 2.6$, we take two cavities

$$\Omega_1 = (-H, -H+\delta) \times (0,1) \quad \text{and} \quad \Omega_2 = (H-\delta, H) \times (0,1) \tag{4.1}$$

with $\delta = 0.125, 0.25, 0.5$ and 1. The distance L between two cavities is given by $L = 2H - 2\delta$. For each test case, the parameter M for the truncated MDtN condition increases from 6 to 12. Figure 2 (a) and (b) show the results for different δ mentioned above when Ω_1 and Ω_2 are triangulated with quadrilateral meshes of $h = 1/400$. For the case of $\delta = 0.125$ we can observe that the errors decay exponentially at the rate of $e^{-\tilde{\mu}_{M+1}\delta}$ (represented by black dash-dot curves) as functions of M, in particular for $6 \le M \le 9$ until reflection errors are dominant compared with finite element errors. We also conduct experiments for fixed $\delta = 0.25$ but with different mesh sizes $h = 1/200, 1/400, 1/800$ and $1/1600$ to see the behaviors of approximation errors in terms of finite element mesh sizes. The results given in Figure 2 (c) and (d) illustrate that as the mesh size is smaller the error plots are closer to the theoretical decay rates (black dash-dot curves). For $\delta = 0.5$ and 1, the reflection errors are small enough even at $M = 6$ compared with finite element errors so that using more Fourier terms in the MDtN condition does not improve the accuracy.

Test (2): Relative L^2-errors in Ω vs. δ. We take the same domain $\widetilde{\Omega}$ as that used for Test (1) with $H = 2.6$. For two cavities Ω_1 and Ω_2 defined by (4.1) with an increasing sequence δ from 0.1 to 0.32 with increment 0.02. Here we consider δ to be the distance from the wave sources to the artificial boundaries Γ_j. We decompose them into quadrilaterals with $h = 1/800$. The distance L between two cavities is given by $L = 2H - 2\delta$ as well. In this test, for fixed $M = 10, 15, 20$ and 25 we compute relative L^2-errors as a function of δ. The resulting plots for $k = 20$ are given in Figure 3. It can be seen that the errors obtained by using the truncated MDtN condition with $M = 10$ decrease exponentially approximately at the rate of $e^{-\tilde{\mu}_{M+1}\delta}$ as δ increases up to 0.2 but they grow after that. Similarly, errors in other cases decrease until some points and then they increase. The reason that the errors grow after δ passes a threshold can be explained in terms of pollution errors of finite element approximations

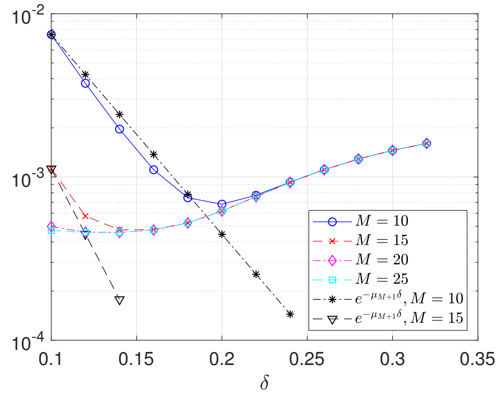


Figure 3: L^2-Errors vs. δ for $k = 20$, $h = 1/800$

for the Helmholtz equation studied in [12, 13, 14]. When the domain size increases along the x_1-axis with both wavenumber k (which is equal to the largest axial frequency μ_0) and mesh size h fixed, after rescaling the computational domain to the unit one we can have $k_s h_s = \textit{constant}$ with increasing rescaled wavenumber k_s and decreasing rescaled mesh size h_s. According to the theory in [12, 13, 14], the condition $k_s h_s = \textit{constant}$ does not give a mesh resolution fine enough to get rid of pollution errors. In fact, the pollution term defined in [12] is proportional to $h_s k_s^2$ and hence once the reflection error determined by the factor $e^{-\tilde{\mu}_{M+1}\delta}$ becomes ignorable compared with finite element errors, overall approximation errors grow with increasing domain size. Therefore it is preferable to set computational domains Ω containing the region of interest as tight as possible and control M to obtain required approximate solutions by choosing $e^{-\tilde{\mu}_{M+1}\delta}$ to be less than a reasonable reflection error. The minimal M in order that $e^{-\tilde{\mu}_{M+1}\delta} < \varepsilon$ for $\delta = 0.1$ is given in Table 1.

k \ ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
6π	8	14	21	28
20	8	14	21	29

Table 1: The minimal M so that $e^{-\tilde{\mu}_{M+1}\delta} < \varepsilon$ for $\delta = 0.1$

Test (3): Relative L^2-errors in Ω vs. L. The purpose of the third test is to show that the proposed technique is independent of the aspect ratio of the removed waveguide Ω_{12}, that is, keeping the height fixed (unit length in this example) it does not depend on the distance L between two artificial boundaries. To this end, let $\delta = 0.2$ be fixed and set $H = L/2 + \delta$ for increasing L from 4 to 42, so that two cavities $\Omega_1 = (-L/2 - \delta, -L/2) \times (0, 1)$ and $\Omega_2 = (L/2, L/2 + \delta) \times (0, 1)$ are L apart from each other. We take $k = 20$ and $M = 14$ for which the reflection error is reduced by the factor $e^{-\tilde{\mu}_{M+1}\delta} < 1.9670 \times 10^{-4}$. The

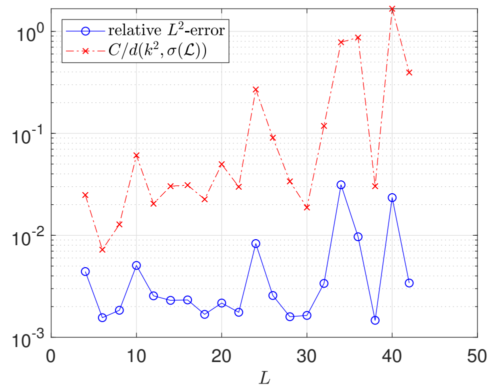


Figure 4: Relative L^2-errors vs. L with $\delta = 0.2$, $M = 14$

errors of finite element approximations for $h = 1/400$ are reported in Figure 4. We can observe that the errors (solid blue line) of most cases except at several peaks are below 3×10^{-3}. In fact, these peaks are related with eigenvalues of the operator $\mathcal{L} = -\Delta$ in $\widetilde{\Omega}$ with the mixed boundary conditions on $\partial\widetilde{\Omega}$. To present the relation, Figure 4 includes the graph of a constant multiple of the resolvent norm given by $1/d(k^2, \sigma(\mathcal{L}))$ with the dash-dot red line, where $\sigma(\mathcal{L})$ is the spectrum of the operator $\mathcal{L}$ and $d(k^2, \sigma(\mathcal{L}))$ is the distance from k^2 to $\sigma(\mathcal{L})$. The shapes of these two plots have a quite good agreement qualitatively. Thus, the performance of the method does not depend on how far away two computational domains are placed if k^2 is away from the spectrum of $\mathcal{L}$.

Test (4): Relative L^2-errors in Ω vs. $\mu_{\min}$. Two cavities Ω_1 and Ω_2 in this test are defined as (4.1) with $H = 2.6$, $\delta = 0.2$ and $L = 2(H - \delta) = 4.8$. We take the wavenumbers $k = k_0 \pm \varepsilon$ for $k_0 = 4\pi$ or 7π with small ε from 10^{-3} to 10^{-8} for which $\mu_{\min} = |k^2 - k_0^2|^{1/2} = |(k_0 \pm \varepsilon)^2 - k_0^2|^{1/2}$. We use $M = 20$ so that $e^{-\mu_{M+1}\delta} \approx 2.3688 \times 10^{-6}$ for $k = 4\pi \pm \varepsilon$ and 3.9568×10^{-6} for $k = 7\pi \pm \varepsilon$. The resulting error plots for $h = 1/400$ are presented in Figure 5, which shows that the performance of the method is independent of near-cutoff modes. As a special case we also conduct a test to show that the decomposition of the right-going and left-going components of the solution and the error is stable with respect to the weighted norm involving $\mu_{\min}$ as in Lemma 2.7 and (3.50). To do this, we choose $k = 7\pi + \varepsilon$ and the exact solution

$$u^{ex}(x_1, x_2) = (e^{\mathrm{i}\mu_7(x_1+L/2)} - e^{-\mathrm{i}\mu_7(x_1+L/2)})Y_7(x_2) = 2\mathrm{i}\sin(\mu_7(x_1 + L/2))Y_7(x_2)$$

with $\lambda_7 = 7\pi$ and $\mu_{\min} = \sqrt{k^2 - \lambda_7^2}$. Here $-L/2$ is the x_1-coordinate of the artificial boundary Γ_1. In this case, we have

$$\|u^{\text{right}}\|_{H^{1/2}(\Gamma_1)} = \|u^{\text{left}}\|_{H^{1/2}(\Gamma_2)} = (1 + (7\pi)^2)^{1/4},$$

whereas

$$\|u^{ex}\|_{H^1(\Omega)} = O(\mu_{\min}) \quad \text{as} \quad \varepsilon \to 0^+.$$

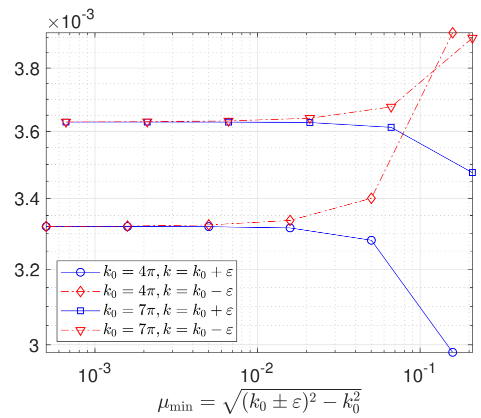


Figure 5: Relative L^2-errors vs. $\mu_{\min}$ for $\varepsilon = 10^{-3}, 10^{-4}, \ldots, 10^{-8}$

We compute the L^2-norms of finite element solutions instead of the full Sobolev norms as it is challenging to compute the fractional Sobolev norms. As shown in Table 2, the right-going and left-going components are bounded with respect to the weighted norm involving $\mu_{\min}$.

ε	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
$\dfrac{\mu_{\min}(\|u^{\text{right}}\|_{L^2(\Gamma_1)} + \|u^{\text{left}}\|_{L^2(\Gamma_2)})}{\|u^{ex}\|_{L^2(\Omega)}}$	1.0094	0.9744	0.9711	0.9707	0.9707	0.9707
$\dfrac{\mu_{\min}(\|e_1^M\|_{L^2(\Gamma_1)} + \|e_2^M\|_{L^2(\Gamma_2)})}{\|e^M\|_{L^2(\Omega)}}$	1.3117	1.2639	1.2592	1.2587	1.2587	1.2587
$\dfrac{\|e^M\|_{L^2(\Omega)}}{\|u^{ex}\|_{L^2(\Omega)}} \quad (\times 10^{-3})$	1.8904	1.8806	1.8796	1.8795	1.8795	1.8795

Table 2: Stable decomposition of the right-going and left-going components

Lastly, as an application of our method, we can consider a wave propagation problem associated with ring resonators. The domain consists of infinite straight waveguide $\mathbb{R} \times (0, 1)$ and three ring resonators R_1, R_2, R_3 centered at $(0, 4), (16, 4)$ and $(32, 4)$, as shown Figure 6 (a) and (c). The outer radius and the inner radius of the ring resonators are 3.5 and 2.5, respectively. For non-reflecting boundary conditions, the perfectly matched layer (PML) is imposed in the regions $(-4, -3) \times (0, 1)$ and $(35, 36) \times (0, 1)$. We compute solutions for $k = 1, 2$ when an incoming wave defined by

$$u^{in}(x_1, x_2) = e^{\mathrm{i}\mu_0(x_1+3)} Y_0(x_2)$$

is propagating through the cross-section at $x_1 = -3$. We confine each resonator in a small computational domain

$$\Omega_1 = R_1 \cup (-4, 3) \times (0, 1), \;\; \Omega_2 = R_2 \cup (13, 19) \times (0, 1), \;\; \Omega_3 = R_3 \cup (29, 36) \times (0, 1)$$

and impose the MDtN condition with $M = 4$ on artificial boundaries. Here Ω_1 and Ω_2 include PML in the left side and in the right side, respectively. The finite element method with $h = 10^{-2}$ produces the solutions shown in Figure 6 (b) and (d). The solutions in Figure 6 (a) and (c) are obtained by solving them in a single computational domain. We can see that two solutions ((a) and (b)), ((c) and (d)) obtained by two different methods coincide with each other .

5. Acknowledgment

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6. References

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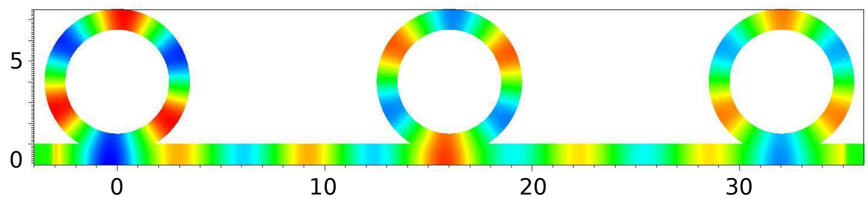
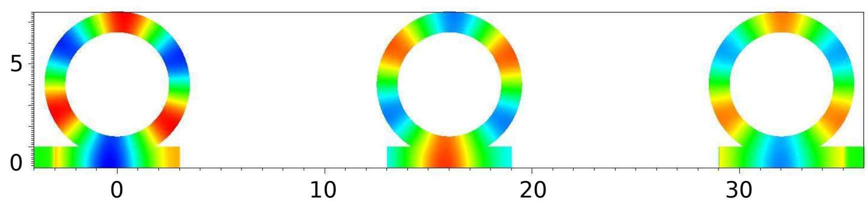
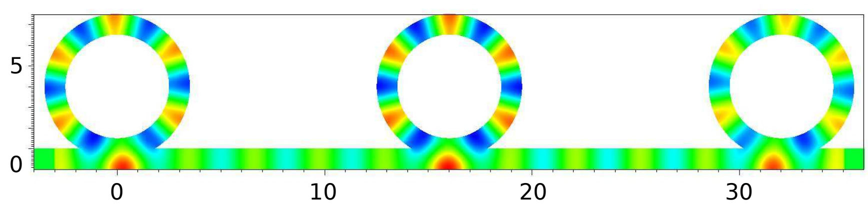


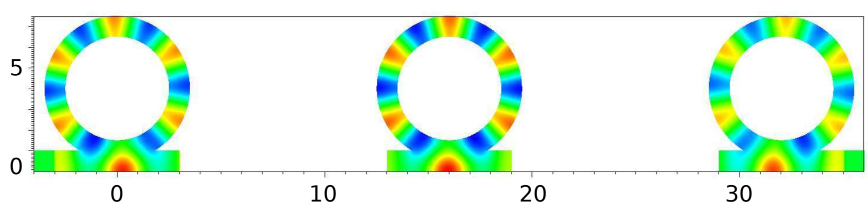
(a) Solution for $k = 1$ obtained in one computational domain



(b) Solution for $k = 1$ obtained by using the MDtN condition



(c) Solution for $k = 2$ obtained in one computational domain



(d) Solution for $k = 2$ obtained by using the MDtN condition

Figure 6: Wave propagation in ring resonators

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