

ANALYSIS OF A CARTESIAN PML APPROXIMATION TO ACOUSTIC SCATTERING PROBLEMS IN \mathbb{R}^2

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ABSTRACT. In this paper, we consider a Cartesian PML approximation to solutions of acoustic scattering problems on an unbounded domain in \mathbb{R}^2 . The perfectly matched layer (PML) technique in a curvilinear coordinate system has been researched for acoustic scattering applications both in theory and computation. Our goal will be to extend the results of spherical/cylindrical PML to PML in Cartesian coordinates, that is, the well-posedness of Cartesian PML approximation on both the unbounded and truncated domains. The exponential convergence of approximate solutions as a function of domain size is also shown. We note that once the stability and convergence of the (continuous) truncated problem has been achieved, the analysis of the resulting finite element approximations is then classical. Finally, the results of numerical computations illustrating the theory and efficiency of the Cartesian PML approach will be given.

1. INTRODUCTION

In this paper we study a Cartesian perfectly matched layer (PML) approximation technique applied to acoustic scattering problems governed by the Helmholtz equation with a real and positive wave number k in two spatial dimension.

For approximation of solutions to acoustic scattering problems posed on an unbounded domain, domain truncation is required. For this purpose many numerical techniques have been designed, for example, boundary integral methods [12, 19, 24], infinite element methods [8, 16] and artificial boundary methods [4, 14, 15, 18, 25]. Here we shall focus on a PML technique based on Cartesian geometry.

A PML is an artificial absorbing layer surrounding the area of computational interest, which is introduced in such a way that it absorbs the scattered waves radiated to the exterior of this region without producing reflected waves. Moreover, a properly constructed PML preserves the solution inside while exponentially damping the solutions within the layer. Because of the exponential decay of PML solutions of the unbounded domain problem, it is natural to truncate the problem to a bounded domain with a convenient boundary condition, e.g., a homogeneous Dirichlet boundary condition, at the outer boundary of the computational domain. This method has

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been applied to the approximation of solutions to Maxwell's equations [5, 6, 9, 10, 13], elasticity problems [11, 21, 20] and acoustic resonances [22, 23, 27] as well as acoustic scattering problems [9, 29].

We consider the exterior Helmholtz problem with Sommerfeld radiation condition,

$$(1.1) \quad \begin{aligned} -\Delta u - k^2 u &= 0 \text{ in } \Omega_c, \\ u &= g \text{ on } \partial\Omega, \\ \lim_{r \rightarrow \infty} r^{1/2} \left| \frac{\partial u}{\partial r} - iku \right| &= 0. \end{aligned}$$

Here k is real and positive and Ω is a bounded domain with a Lipschitz continuous boundary contained in the square* $[-a, a]^2$ for some positive a and Ω_c denotes $\mathbb{R}^2 \setminus \bar{\Omega}$.

The simplest example of a Cartesian PML approximation involves an even function $\tilde{\sigma}$ satisfying

$$(1.2) \quad \begin{aligned} \tilde{\sigma}(x) &= 0 \text{ for } |x| \leq a, \\ \tilde{\sigma}(x) &: \text{increasing for } a < x < b, \\ \tilde{\sigma}(x) &= \sigma_0 \text{ for } |x| \geq b. \end{aligned}$$

Here $0 < a < b$ and $\sigma_0 > 0$ is a parameter (the PML strength). Then the coordinate stretching \tilde{x} is defined by

$$\tilde{x} = x(1 + i\tilde{\sigma}(x)) \text{ for } x \in \mathbb{R},$$

where the imaginary part $x\tilde{\sigma}(x)$ is responsible for attenuating scattering waves exponentially in the perfectly matched layer. For our analysis, we require that

$$(1.3) \quad \begin{aligned} \tilde{\sigma} &\in C^1(\mathbb{R}) \text{ with } \tilde{\sigma} \text{ piecewise smooth and} \\ \tilde{\sigma}'' &\in L^\infty(\mathbb{R}). \end{aligned}$$

The requirement that $\tilde{\sigma}$ be constant for $|x| \geq b$ is necessary for the application of the results in [28]. We believe that the last condition can be removed and its removal is the topic of work in preparation.

There are many other different coordinate stretching functions used elsewhere. These are equivalently presented in terms of σ . In this case the stretching is taken to be

$$\tilde{x} = x + i \int_0^x \sigma(t) dt.$$

This, of course, is equivalent to taking

$$\tilde{\sigma}(x) = x^{-1} \int_0^x \sigma(t) dt$$

*We consider a domain in \mathbb{R}^2 for convenience. The extension to domains in \mathbb{R}^3 is completely analogous. The modification required for the 3 dimensional analysis will be discussed in Appendix B

above. Piecewise constant, power functions or unbounded functions have been proposed for σ [20, 22, 13, 7]. To fit into the analysis presented here (see Remark 3.5), we need to explicitly exclude the case of piecewise constant σ .

The PML reformulation leads to the study of a source problem: for $f \in L^2(\Omega_c)$, find $\hat{u} \in H_0^1(\Omega_c)$ satisfying

$$(1.4) \quad A(\hat{u}, \phi) - k^2(J(x)\hat{u}, \phi) = (J(x)f, \phi) \text{ for all } \phi \in H_0^1(\Omega_c).$$

Here $H_0^1(\Omega_c)$ denotes the Sobolev space of order one on Ω_c consisting of complex valued functions which vanish on $\partial\Omega$, $J(x) = d(x_1)d(x_2)$, $d(x) = 1 + i(x\tilde{\sigma}(x))'$ and

$$(1.5) \quad \begin{aligned} A(u, v) &= \int_{\Omega_c} \left[\frac{d(x_2)}{d(x_1)} \frac{\partial u}{\partial x_1} \frac{\partial \bar{v}}{\partial x_1} + \frac{d(x_1)}{d(x_2)} \frac{\partial u}{\partial x_2} \frac{\partial \bar{v}}{\partial x_2} \right] dx, \\ (f, g) &= \int_{\Omega_c} f \bar{g} dx. \end{aligned}$$

In [9], an analysis of the source problem on the infinite domain with spherical PML was given by first showing that the resulting form was coercive up to a lower order perturbation on a bounded domain. A standard argument by compact perturbation [32, 35] then shows stability of the source problem once uniqueness has been established. Unfortunately, this approach fails for Cartesian PML. The problem is, e.g., that the coefficient of the x_1 derivatives in $A(\cdot, \cdot)$ equals $-k^{-2}$ times the coefficient of the lower order contribution on the left hand side of (1.4) when $x_1 \in (-a, a)$. As $\Omega_c \cap ((-a, a) \times \mathbb{R})$ is an unbounded domain, we cannot restore coercivity by a lower order perturbation on a BOUNDED domain.

In [28], we examined the essential spectrum of the Cartesian PML operator corresponding to the scattering problem on the full domain Ω_c . There we showed that any point on the real axis excluding the origin is either in the resolvent set or is in the discrete spectrum (i.e., an isolated point of the spectrum of finite algebraic multiplicity). The first result of this paper is to show a uniqueness result for this problem for all real $k \neq 0$. Using this and the results of [28], we conclude stability of the PML scattering problem on the infinite domain. This is one of the main ingredients in our subsequent analysis of the truncated Cartesian PML problem.

Another critical ingredient for the analysis in this paper is the construction of solutions to the PML equation (and the original equation) in terms of integrals of the function and its normal derivative against a “fundamental solution” and its normal derivative. In the case of the Helmholtz equation with a real and positive wave number k , these results are classical. These results are mentioned for the PML Helmholtz equation based on a smooth convex geometry by Lassas and Somersalo [30]. We include a proof via analytic continuation for completeness. Such results are needed for proving uniqueness and exponential decay of the PML problem on the infinite domain, a fundamental part of this paper.

The goal of this paper is to provide results that are uniform in δ (the size of the computational domain) as δ becomes large. Accordingly, we shall use C , with or without subscript, to denote a generic positive constant which can be taken independent of δ with the restriction that $\delta \geq \delta_0$ for a fixed δ_0 . The only exception to this appears in the finite element section (Section 6) where we have not made an attempt to estimate an elliptic regularity constant.

The remainder of this paper is organized as follows. In Section 2, we reformulate a model problem with a Cartesian PML and introduce certain coefficient functions for the Cartesian PML operator. We study the fundamental solution to the Cartesian PML equation and its exponential decay in Section 3. In Section 4, we show the existence and uniqueness of the Cartesian PML equation with a real and positive wave number k in the infinite domain. Section 5 shows the truncated PML problem is well-posed provided that the computational domain is large enough. In this case, we show that the corresponding solution converges exponentially to that of the infinite domain problem with increasing domain size. In Section 6 the solvability of the finite element problem is studied and the approximation error is analyzed. Finally, Section 7 provides the results of numerical experiments illustrating the convergence of finite element PML approximations.

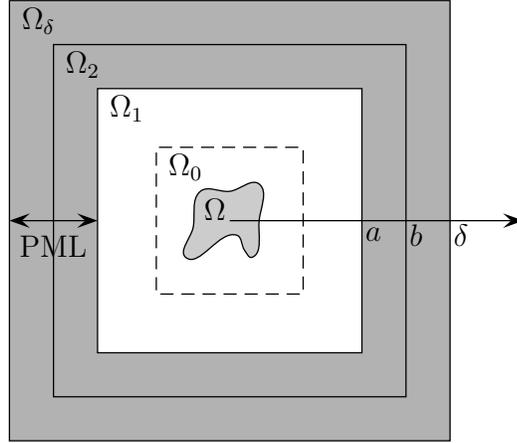
2. CARTESIAN PML REFORMULATION

We start with a remark concerning a slightly more general PML formulation.

Remark 2.1. In the introduction, we considered a simple PML example where the same stretching function was used in each direction. In an application where the domain more naturally fits into a rectangle $[-a_1, a_1] \times [-a_2, a_2]$, it is more reasonable (and computationally efficient) to use direction dependent PML stretching functions. For example, we use even functions $\tilde{\sigma}_j$, for $j = 1, 2$ satisfying (1.2) with a, b and σ_0 replaced by a_j, b_j and σ_{j0} , respectively. The only changes in (1.4) and (1.5) involve replacement of $d(x_j)$ by $d_j(x_j) \equiv 1 + i(x_j \tilde{\sigma}_j(x_j))'$. As the analysis presented below is identical[†] for direction dependent PML stretching, for convenience of notation, from here on, we shall revert back to the case of the introduction, i.e., $\tilde{\sigma}_1 = \tilde{\sigma}_2 = \tilde{\sigma}$.

We shall use a sequence of the strictly increasing square domains, $\Omega_1 = (-a, a)^2$, $\Omega_2 = (-b, b)^2$ (a and b as in the introduction) and $\Omega_\delta = (-\delta, \delta)^2$ such that $\Omega \subset \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \Omega_\delta$ (see Figure 1). Here Ω_0 is an auxiliary square domain between Ω and Ω_1 . Let Γ_j denote the boundary of Ω_j for $j = 0, 1, 2$ and δ . In particular, as we shall see, the infinite domain PML model preserves the solution of (1.1) in Ω_1 . The domain $\Omega_\delta \setminus \bar{\Omega}$ is the domain of numerical computation. Here we assume that the origin is inside the scatterer Ω and the sides of square domains are parallel to the coordinate axes.

[†]Except possibly replacing squares with suitable rectangles.


 FIGURE 1. PML domains in \mathbb{R}^2

The Cartesian PML technique can be thought of as a formal complex shift in the Cartesian coordinate system. We shall use the following notations:

$$\begin{aligned}
 \tilde{d}(x) &\equiv 1 + i\tilde{\sigma}(x), \\
 \tilde{x}(x) &\equiv x(1 + i\tilde{\sigma}(x)) = x\tilde{d}(x), \\
 \sigma(x) &\equiv \frac{d}{dx}(x\tilde{\sigma}(x)) = \tilde{\sigma}(x) + x\frac{d\tilde{\sigma}(x)}{dx}, \\
 (2.1) \quad d(x) &\equiv \frac{d\tilde{x}}{dx} = 1 + i\sigma(x) \quad \text{for } x \in \mathbb{R} \text{ and,} \\
 J(x) &\equiv d(x_1)d(x_2), \\
 H(x) &\equiv \begin{bmatrix} d(x_2)/d(x_1) & 0 \\ 0 & d(x_1)/d(x_2) \end{bmatrix} \text{ for } x = (x_1, x_2) \in \mathbb{R}^2.
 \end{aligned}$$

The goal of this paper is to study the weak form (1.4) and its analogue on the truncated domain. For the truncated problem, the only change in (1.4) and (1.5) is that Ω_c and $H_0^1(\Omega_c)$ are replaced by $\Omega_\delta \setminus \bar{\Omega}$ and $H_0^1(\Omega_\delta \setminus \bar{\Omega})$, respectively.

3. FUNDAMENTAL SOLUTION TO THE CARTESIAN PML HELMHOLTZ EQUATION

In this section, we investigate the fundamental solution to the Cartesian PML equation in \mathbb{R}^2 . For this, we shall generalize the complex stretching functions. Let σ_M denote the maximum of σ , and

$$U \equiv \{z \in \mathbb{C} : \operatorname{Re}(z) > -1/(2\sigma_M)\}.$$

For $z \in U$, we define $\tilde{d}^z, \tilde{x}^z, \sigma^z, d^z, J^z$ in (2.1) with z in place of i . We also introduce a “stretched” differential operator $\tilde{\Delta}^z$ given by

$$\tilde{\Delta}^z = \frac{1}{d^z(x_1)} \frac{\partial}{\partial x_1} \left(\frac{1}{d^z(x_1)} \frac{\partial}{\partial x_1} \right) + \frac{1}{d^z(x_2)} \frac{\partial}{\partial x_2} \left(\frac{1}{d^z(x_2)} \frac{\partial}{\partial x_2} \right).$$

Finally, we denote $\tilde{x}_j^z = \tilde{x}^z(x_j)$ and define a complexified distance between $(\tilde{x}_1^z, \tilde{x}_2^z)$ and $(\tilde{y}_1^z, \tilde{y}_2^z) = (\tilde{x}^z(y_1), \tilde{x}^z(y_2))$ by

$$\tilde{r}^z = \sqrt{(\tilde{x}_1^z - \tilde{y}_1^z)^2 + (\tilde{x}_2^z - \tilde{y}_2^z)^2}.$$

In the case of $z = i$, we will also use the notation \tilde{r} without z dependency. The properties of \tilde{r}^z are described in the following lemmas. Their proofs will be provided in Appendix A.

In the definition of \tilde{r}^z , we use the square root with a branch cut on the negative real axis and note the following lemma.

Lemma 3.1. *For $z \in U$ there exists $\varepsilon > 0$ such that for $x \neq y$,*

$$-\pi + \varepsilon \leq \arg((\tilde{x}_1^z - \tilde{y}_1^z)^2 + (\tilde{x}_2^z - \tilde{y}_2^z)^2) \leq \pi - \varepsilon.$$

The constant ε appearing above depends on $|\operatorname{Im}(z)|$ and hence holds uniformly on subsets

$$U_\beta \equiv \{z \in U \text{ with } |\operatorname{Im}(z)| \leq \beta\},$$

i.e., $\varepsilon = \varepsilon(\beta)$ on U_β .

Remark 3.2. In the proof of Lemma 3.1, we actually show that for $\operatorname{Im}(z) > 0$,

$$0 \leq \arg((\tilde{x}_1^z - \tilde{y}_1^z)^2 + (\tilde{x}_2^z - \tilde{y}_2^z)^2) \leq \pi - \varepsilon.$$

Lemma 3.3. *For $z \in U$ and $x, y \in \mathbb{R}^2$, there exist positive constants C_1 and C_2 depending on z such that*

$$(3.1) \quad C_1|x - y| \leq |\tilde{r}^z| \leq C_2|x - y|.$$

Moreover, the constants $C_1 = C_1(\alpha)$ and $C_2 = C_2(\alpha)$ can be chosen independent of $z \in U$ provided that $|z| \leq \alpha$.

The fundamental solution to the Helmholtz equation in \mathbb{R}^2 satisfying the Sommerfeld radiation condition at infinity with k real and positive is $\Phi(r) = \frac{i}{4}H_0^1(kr)$. We have

$$(3.2) \quad \int_{\mathbb{R}^2} (-\Delta_y + k^2 I)u(y)\Phi(|x - y|) dy = u(x) \text{ for } u \in C_0^\infty(\mathbb{R}^2).$$

Here $H_0^1 = J_0 + iY_0$ is the Hankel function of the first kind of zero order and J_0 and Y_0 are the Bessel functions of the first and second kind, respectively. We have

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(k!)^2},$$

and for $z \in \mathbb{C} \setminus (-\infty, 0]$

$$Y_0(z) = \frac{2}{\pi} J_0(z) \ln \frac{z}{2} + W_0(z)$$

for an entire function $W_0(z)$ with $\lim_{z \rightarrow 0} W_0(z) = 2\gamma/\pi$, where $\gamma = 0.57721566 \dots$ is Euler's constant (See, e.g., [31]). Thus

$$\begin{aligned} H_0^1(z) &= \frac{2i}{\pi} J_0(z) \ln \frac{z}{2} + E(z), \\ H_0^{1'}(z) &= \frac{2i}{\pi} \left(J_0'(z) \ln \frac{z}{2} + J_0(z) \frac{1}{z} \right) + E'(z) \end{aligned}$$

for an entire function $E(z)$ with $\lim_{z \rightarrow 0} E(z) = 1 + 2\gamma i/\pi$. It follows that there exist $C_b > 0$ and $r_b > 0$ such that

$$(3.3) \quad \begin{aligned} |\Phi(z)| &\leq C_b |\ln |z||, \\ |\Phi'(z)| &\leq \frac{C_b}{|z|} \end{aligned}$$

on $B(0, r_b) \setminus (-r_b, 0]$. Here $B(0, r_b) \subset \mathbb{C}$ is a ball of radius r_b centered at $z = 0$. On the other hand, for large $|z|$, we have

$$(3.4) \quad \begin{aligned} H_0^1(z) &= \left(\frac{2}{\pi z} \right)^{1/2} e^{i(z-\pi/4)} \left(1 + O\left(\frac{1}{z} \right) \right) \text{ for } |\arg(z)| \leq \pi - \varepsilon, \\ H_0^{1'}(z) &= \left(\frac{2}{\pi z} \right)^{1/2} e^{i(z+\pi/4)} \left(1 + O\left(\frac{1}{z} \right) \right) \text{ for } |\arg(z)| \leq \pi - \varepsilon \end{aligned}$$

with arbitrary small ε [1, 31].

Theorem 3.4. *Assume that $z \in U$. Then $\tilde{\Phi}^z(x, y) \equiv J^z(y)\Phi(\tilde{r}^z)$ satisfies*

$$(3.5) \quad u(x) = \int_{\mathbb{R}^2} (-\tilde{\Delta}_y^z + k^2 I)u(y) \tilde{\Phi}^z(x, y) dy$$

for all $u \in C_0^\infty(\mathbb{R}^2)$.

Remark 3.5. The above theorem is a fundamental step in the analysis of this paper. The smoothness assumptions on the PML function $\tilde{\sigma}$ are sufficient to make the operator $\tilde{\Delta}_y^z$ well defined on smooth functions. In the case of discontinuous σ , the operator $\tilde{\Delta}_y^z$ would only make sense on functions whose derivatives satisfied z -dependent jump conditions at the lines of discontinuity. The proof of this theorem and subsequent results, e.g., Remark 4.2, would not hold in this case.

To prove (3.5), for $u \in C_0^\infty(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, we define

$$F(z) = \int_{\mathbb{R}^2} (-\tilde{\Delta}_y^z + k^2 I)u(y) \tilde{\Phi}^z(x, y) dy = \int_{\mathbb{R}^2} J^z(y) (-\tilde{\Delta}_y^z + k^2 I)u(y) \Phi(\tilde{r}^z) dy$$

and we shall show that $F(z)$ is analytic on U . This will be treated in the following lemmas.

First, we need to justify that the integral in (3.5) is well defined. To this end, fix $x \in \mathbb{R}^2$, set $P(y, z) = J^z(y)(\tilde{\Delta}_y^z + k^2 I)u(y)$ and $G(y, z) = P(y, z)\Phi(\tilde{r}^z)$. For any $z_0 \in U$ there exists $\epsilon > 0$ such that $\tilde{B}(z_0, \epsilon) \subset U$.

Lemma 3.6. *Let z_0, ϵ and $G(y, z)$ be defined as above. Then, $G(\cdot, z)$ and $\frac{\partial}{\partial z}G(\cdot, z)$ are integrable for each $z \in B(z_0, \epsilon)$. In addition, there exists an integrable function $\mathcal{G}(y)$ such that*

$$(3.6) \quad \left| \frac{\partial}{\partial z}G(y, z) \right| \leq \mathcal{G}(y) \text{ for all } z \in B(z_0, \epsilon) \text{ and } y \neq x.$$

Proof. Note that $\Phi(\tilde{r}^z)$ is a continuous function of y except at $y = x$. By Lemma 3.3, there exists $s > 0$ such that $|\tilde{r}^z| < r_b$ for $(y, z) \in B(x, s) \times B(z_0, \epsilon)$. It follows from Lemma 3.1, (3.3) and Lemma 3.3 that there exists a constant $C_{sing} > 0$ such that

$$(3.7) \quad \begin{aligned} |\Phi(\tilde{r}^z)| &\leq C_b |\ln |\tilde{r}^z|| \leq C_{sing} |\ln |x - y||, \\ |\Phi'(\tilde{r}^z)| &\leq \frac{C_b}{|\tilde{r}^z|} \leq \frac{C_{sing}}{|x - y|} \end{aligned}$$

for $(y, z) \in \tilde{B}(x, s) \times B(z_0, \epsilon)$. Here $\tilde{B}(x, s)$ denotes $B(x, s) \setminus \{x\}$

Moreover, by the assumption on $\tilde{\sigma}$, (1.3),

$$(3.8) \quad |P(y, z)|, \left| \frac{\partial}{\partial z}P(y, z) \right| \leq C_p (|\Delta u(y)| + |\nabla u(y)| + |u(y)|) \text{ for all } y \in \mathbb{R}^2$$

with C_p independent of $z \in B(z_0, \epsilon)$.

By (3.7)-(3.8), $G(\cdot, z)$ is integrable on the neighborhood $B(x, s)$ for all $z \in B(z_0, \epsilon)$. Its integrability outside of $B(x, s)$ follows from (3.8) and the fact that u is compactly supported (since $\Phi(\tilde{r}^z)$ is bounded on $\text{supp}(u) \setminus B(x, s)$).

For the derivative

$$(3.9) \quad \begin{aligned} \frac{\partial}{\partial z}G(y, z) &= \left(\frac{\partial}{\partial z}P(y, z) \right) \Phi(\tilde{r}^z) + P(y, z) \frac{\partial}{\partial z}\Phi(\tilde{r}^z) \\ &= \left(\frac{\partial}{\partial z}P(y, z) \right) \Phi(\tilde{r}^z) + P(y, z) \Phi'(\tilde{r}^z) \frac{\partial \tilde{r}^z}{\partial z}. \end{aligned}$$

Except for the derivative of \tilde{r}^z with respect to z , the functions in (3.9) are estimated as above.

For $\partial \tilde{r}^z / \partial z$, we observe

$$x_j \tilde{\sigma}(x_j) - y_j \tilde{\sigma}(y_j) = \sigma(\xi_j)(x_j - y_j)$$

for ξ_j between x_j and y_j . Thus for $z \in B(z_0, \epsilon)$,

$$(3.10) \quad \begin{aligned} \left| \frac{\partial \tilde{r}^z}{\partial z} \right| &= \left| \frac{\sum_{j=1,2} (\tilde{x}_j - \tilde{y}_j) (x_j \tilde{\sigma}(x_j) - y_j \tilde{\sigma}(y_j))}{\tilde{r}^z} \right| \\ &= \left| \frac{\sum_{j=1,2} (x_j - y_j)^2 (1 + z\sigma(\xi_j)) \sigma(\xi_j)}{\tilde{r}^z} \right| \leq C_r |x - y|, \end{aligned}$$

where we used Lemma 3.3.

Let $h(y)$ be defined by

$$h(y) = \begin{cases} \frac{C_{sing}}{|x - y|} & \text{for } y \in \tilde{B}(x, s) \\ C_{sup} & \text{for } y \in \mathbb{R}^2 \setminus B(x, s), \end{cases}$$

where C_{sup} is the supremum of $|\Phi(\tilde{r}^z)|$ and $|\Phi'(\tilde{r}^z)|$ for $y \in \text{supp}(u) \setminus B(x, s)$ and $z \in B(z_0, \epsilon)$. Since $|\ln|x - y|| \leq 1/|x - y|$ for $|x - y| < s < 1$,

$$|\Phi(\tilde{r}^z)|, |\Phi'(\tilde{r}^z)| \leq h(y) \text{ on } \text{supp}(u).$$

Then applying (3.7), (3.8) and (3.10) to (3.9) gives

$$\left| \frac{\partial}{\partial z} G(y, z) \right| \leq C_p (|\Delta u(y)| + |\nabla u(y)| + |u(y)|) h(y) (1 + C_r |x - y|)$$

and (3.6) follows. This completes the proof. \square

Lemma 3.7. *For $u \in C_0^\infty(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, $F(z)$ defined as above is analytic on U .*

Proof. For $z_0 \in U$ choose ϵ as in Lemma 3.6. It suffices to show that the limit of $(F(z+h) - F(z))/h$ as $h \rightarrow 0$ exists for $z \in B(z_0, \epsilon)$. This, in turn, will follow by dominated convergence once we show that there exists an integrable function $\tilde{\mathcal{G}}(y)$ such that

$$\left| \frac{G(y, z+h) - G(y, z)}{h} \right| \leq \tilde{\mathcal{G}}(y).$$

Then,

$$\begin{aligned} \frac{dF}{dz} &= \int_{\mathbb{R}^2} \lim_{h \rightarrow 0} \frac{G(y, z+h) - G(y, z)}{h} dy \\ &= \int_{\mathbb{R}^2} \frac{\partial}{\partial z} G(y, z) dy. \end{aligned}$$

By applying the mean value theorem and the Cauchy-Riemann equations, it is easy to show that for an analytic function w ,

$$|w(z_1) - w(z_2)| \leq 2|z_1 - z_2| \max_{\alpha \in (0,1)} \left| \frac{dw}{dz}(\alpha z_1 + (1-\alpha)z_2) \right|.$$

Thus, by (3.6),

$$\left| \frac{G(y, z+h) - G(y, z)}{h} \right| < 2\mathcal{G}(y) \text{ for } z \in B(z_0, \epsilon),$$

which completes the proof. \square

Proof of Theorem 3.4. First, we will prove (3.5) for real $z \in U$. In this case the mapping $y \mapsto \tilde{y}^z$ is a diffeomorphism of \mathbb{R}^2 with the Jacobian $J^z(y)$ and \tilde{r}^z is $|\tilde{x}^z - \tilde{y}^z|$, l^2 -norm of $\tilde{x}^z - \tilde{y}^z$ in \mathbb{R}^2 . Let $u \in C_0^\infty(\mathbb{R}^2)$ and define $v(\tilde{y}^z) \equiv u(y)$. By change of variables and (3.2),

$$\begin{aligned} F(z) &= \int_{\mathbb{R}^2} J^z(y) (-\tilde{\Delta}_y^z + k^2 I) u(y) \Phi(|\tilde{x}^z - \tilde{y}^z|) dy \\ &= \int_{\mathbb{R}^2} (-\Delta_{\tilde{y}^z} + k^2 I) v(\tilde{y}^z) \Phi(|\tilde{x}^z - \tilde{y}^z|) d\tilde{y}^z \\ &= v(\tilde{x}^z) = u(x), \end{aligned}$$

which means that $F(z)$ is constant on $U \cap \mathbb{R}$. Since $F(z)$ is analytic on U by Lemma 3.7 and constant on $U \cap \mathbb{R}$, $F(z)$ must be constant. Therefore $F(z) = u(x)$ for all $z \in U$. \square

Remark 3.8. The formula (3.5) can be extended to $u \in H^2(\mathbb{R}^2)$. To see this, we first note that for any x , $\tilde{\Phi}^z(x, \cdot)$ is in $L^2(\mathbb{R}^2)$. In fact, this follows from the fact that $\tilde{\Phi}^z(x, y)$ has an L^2 -integrable singularity at $y = x$ (by (3.7)) and decays exponentially at infinity (by Lemma 4.1 and (3.4)) (we leave the details as an exercise for the reader).

We then consider, for fixed x , the integral operator $I : L^2(\mathbb{R}^2) \rightarrow \mathbb{R}$ defined by

$$I(f) = \int_{\mathbb{R}^2} f(y) \tilde{\Phi}^z(x, y) dy.$$

The Schwarz inequality,

$$|I(f)| \leq \|f\|_{L^2(\mathbb{R}^2)} \|\tilde{\Phi}^z(x, \cdot)\|_{L^2(\mathbb{R}^2)},$$

implies I is bounded. Now, for $u \in H^2(\mathbb{R}^2)$, there exists $u_n \in C_0^\infty(\mathbb{R}^2)$ converging to u in $H^2(\mathbb{R}^2)$. Then $u_n(x) = I(-(\tilde{\Delta}^z + k^2 I)u_n)$ converges to $I(-(\tilde{\Delta}^z + k^2 I)u)$. On the other hand, a Sobolev imbedding theorem implies that $u_n(x)$ converges to $u(x)$, i.e., $u(x) = I(-(\tilde{\Delta}^z + k^2 I)u)$ for all $u \in H^2(\mathbb{R}^2)$.

For each $x \in \mathbb{R}^2$, the function $\Phi(\tilde{r}^z)$ (as a function of y) satisfies the Cartesian PML Helmholtz equation as noted in the following lemma.

Lemma 3.9. *Assume that $y \neq x$ in \mathbb{R}^2 and $z \in U$. Then*

$$(\tilde{\Delta}_y^z + k^2 I)\Phi(\tilde{r}^z) = 0.$$

Proof. Let x, y, z be as above. We note that

$$(3.11) \quad \tilde{F}(z) \equiv (\tilde{\Delta}_y^z + k^2 I)\Phi(\tilde{r}^z) = \Phi''(\tilde{r}^z) + \frac{1}{\tilde{r}^z}\Phi'(\tilde{r}^z) + k^2\Phi(\tilde{r}^z).$$

As $\tilde{F}(z)$ is analytic on U and vanishes for real $z \in U$, $\tilde{F}(z)$ vanishes identically. \square

4. SOLVABILITY OF THE PML PROBLEM IN THE INFINITE DOMAIN

From this section on, we take $z = i$ and z -dependency in notations will be omitted for simplicity. Also, C and α represent generic constants which do not depend on δ .

We start with a lemma involving the imaginary part of the stretched radius \tilde{r} .

Lemma 4.1. *There is a positive constant α such that for $y \in [-a, a]^2$ and $\|x\|_\infty \geq b$,*

$$(4.1) \quad \text{Im}(\tilde{r}) \geq \alpha|x|.$$

In addition, (4.1) holds also if $y \in [-m, m]^2$, $\|x\|_\infty = R \geq 2m$ and $m \geq b$.

Proof. Let y be in $[-a, a]^2$ and x be in the complement of $(-b, b)^2$. Assume without loss of generality that $|x_1| = \|x\|_\infty$. Then

$$(4.2) \quad \begin{aligned} \tilde{r}^2 &= (x_1 - y_1)^2 - (\sigma_0 x_1)^2 + 2(x_1 - y_1)\sigma_0 x_1 i \\ &\quad + (x_2 - y_2)^2 - (\tilde{\sigma}(x_2)x_2)^2 + 2(x_2 - y_2)\tilde{\sigma}(x_2)x_2 i \\ &\equiv R_1 + I_1 i + R_2 + I_2 i \equiv R_3 + I_3 i. \end{aligned}$$

Now $I_1 > 0$ and $I_2 \geq 0$ and there is a positive constant c_1 satisfying

$$(4.3) \quad 2\text{Re}(\tilde{r})\text{Im}(\tilde{r}) = I_3 \geq I_1 \geq c_1\|x\|_\infty^2.$$

Moreover, Remark 3.2 implies that the real part of \tilde{r} is non-negative, and using Lemma 3.3

$$(4.4) \quad \text{Re}(\tilde{r}) \leq |\tilde{r}| \leq c_2\|x\|_\infty.$$

An elementary calculation using (4.3) and (4.4) gives

$$\text{Im}(\tilde{r}) \geq \frac{c_1}{2c_2}\|x\|_\infty \geq \frac{c_1}{2\sqrt{2}c_2}|x|.$$

For the second case, we start with (for $j = 1, 2$)

$$\tilde{x}_j - \tilde{y}_j = (x_j - y_j) + (\tilde{\sigma}(x_j)x_j - \tilde{\sigma}(y_j)y_j)i.$$

Now,

$$(4.5) \quad \tilde{\sigma}(x_j)x_j - \tilde{\sigma}(y_j)y_j = \int_{y_j}^{x_j} \sigma(s) ds = \sigma(\zeta_j)(x_j - y_j)$$

for some ζ_j between x_j and y_j . Assume without loss of generality that $|x_1| = \|x\|_\infty$. We expand \tilde{r}^2 analogous to (4.2), i.e.,

$$\tilde{r}^2 \equiv R_1 + I_1 i + R_2 + I_2 i \equiv R_3 + I_3 i.$$

Now, (4.5) and the fact that $\sigma \geq 0$ implies that $I_2 \geq 0$. Moreover, the integral representation of the difference in (4.5) implies that if $x_1 \geq 2m$,

$$\int_{y_1}^{x_1} \sigma(s) ds \geq \sigma_0(x_1 - b) \geq \frac{\sigma_0}{3}(x_1 - y_1) > 0.$$

Thus

$$I_1 \geq \frac{2\sigma_0}{3}(x_1 - y_1)^2 \geq \frac{\sigma_0}{3}\|x\|_\infty^2.$$

The same argument implies the above inequality when $x_1 < 0$. Thus, (4.3) and (4.4) follow for this case as well, and the conclusion of the lemma immediately follows as above. \square

Remark 4.2. Let D be a domain in \mathbb{R}^2 . Let \tilde{D} be a domain containing \bar{D} with $\beta = \text{dist}(D, \partial\tilde{D})$ for a fixed constant β and suppose that $u \in H^1(\tilde{D})$ satisfies

$$A(u, \phi) - k^2(J(x)u, \phi) = 0 \text{ for all } \phi \in H_0^1(\tilde{D}).$$

Then, by interior regularity (note that the assumption on $\tilde{\sigma}$ (1.3) is necessary for the interior regularity, see, e.g., [17]), u is in $H^2(D)$ and there is a constant C depending on β , k and σ (but not on δ even if D depends on δ) satisfying

$$\|u\|_{H^2(D)} \leq C\|u\|_{L^2(\tilde{D})}.$$

This is because the coefficients defining $A(\cdot, \cdot)$ satisfy

$$\text{Re}(d(x)/d(y)) \geq \alpha_0 \text{ and } \text{Re}(d(y)/d(x)) \geq \alpha_0$$

for a positive number α_0 and all $x, y \in \mathbb{R}$. This, in turn, implies the coercivity of $A(\cdot, \cdot) + (\cdot, \cdot)$ and the interior regularity result follows.

We first derive an integral formula for solutions to $(\tilde{\Delta} + k^2I)u = 0$ on Ω_c .

Theorem 4.3. *Assume that $u \in H^1(\Omega_c)$ satisfies $(\tilde{\Delta} + k^2I)u = 0$ on Ω_c . Then, for $x \in \mathbb{R}^2 \setminus \bar{\Omega}_0$,*

$$(4.6) \quad u(x) = \int_{\Gamma_0} \left[u(y) \frac{\partial \Phi(\tilde{r})}{\partial n_y} - \Phi(\tilde{r}) \frac{\partial u}{\partial n}(y) \right] dS_y,$$

where n is the outward unit normal vector on Γ_0 .

Proof. We verify the theorem for $x \in [-m, m]^2$ with $m \geq b$. Let Ω_R be a square domain $(-R, R)^2$ with $R \geq 2m$ and Γ_R its boundary. Let $D = \Omega_R \setminus \bar{\Omega}_0$. Since u is in $H_{loc}^2(\Omega_c)$, u is in H^2 on a neighborhood \tilde{D} of D (see Figure 2). Using a cutoff function, which is one on D and supported on \tilde{D} , we can define a compactly supported

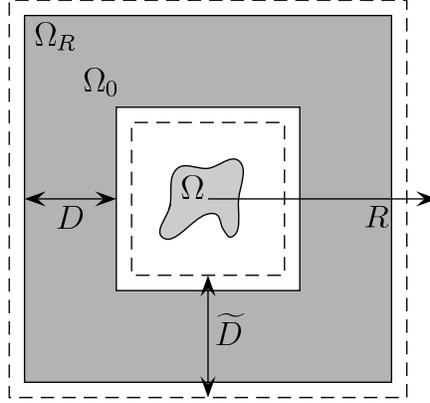


FIGURE 2. Ω_R and Ω_0 are two square domains, and Γ_R and Γ_0 denote their boundary, respectively. $D = \Omega_R \setminus \bar{\Omega}_0$ and \tilde{D} is a domain containing \bar{D} .

extension \tilde{u} in $H^2(\mathbb{R}^2)$ of u defined on D . For $x \in D$ it follows from Theorem 3.4 and Remark 3.8 that

$$\begin{aligned} -u(x) &= \int_{\mathbb{R}^2} ((\tilde{\Delta}_y + k^2 I)\tilde{u}(y))\tilde{\Phi}(x, y) dy \\ &= \int_{\Omega_0} ((\tilde{\Delta}_y + k^2 I)\tilde{u}(y))\tilde{\Phi}(x, y) dy + \int_{\mathbb{R}^2 \setminus \bar{\Omega}_R} ((\tilde{\Delta}_y + k^2 I)\tilde{u}(y))\tilde{\Phi}(x, y) dy. \end{aligned}$$

By integration by parts and Lemma 3.9

$$\begin{aligned} u(x) &= - \int_{\Gamma_0} [\Phi(\tilde{r})n^t H \nabla u(y) - u(y)n^t H \nabla \Phi(\tilde{r})] dS_y \\ &\quad + \int_{\Gamma_R} [\Phi(\tilde{r})n^t H \nabla u(y) - u(y)n^t H \nabla \Phi(\tilde{r})] dS_y, \end{aligned}$$

where n is the outward unit normal vector on the boundaries of Ω_0 and Ω_R .

Since $|d(y_j)|$ for $j = 1, 2$ is bounded above and below away from zero, by a Schwarz inequality

$$\begin{aligned} I &\equiv \left| \int_{\Gamma_R} [\Phi(\tilde{r})n^t H \nabla u(y) - u(y)n^t H \nabla \Phi(\tilde{r})] dS_y \right| \\ &\leq C (\|\Phi(\tilde{r})\|_{L^2(\Gamma_R)} \|\nabla u\|_{L^2(\Gamma_R)} + \|u\|_{L^2(\Gamma_R)} \|\nabla \Phi(\tilde{r})\|_{L^2(\Gamma_R)}). \end{aligned}$$

Set $S_\gamma = \{x \in \mathbb{R}^2 : \text{dist}(x, \Gamma_R) < \gamma\}$ with γ independent of R and small enough so that $S_\gamma \subset \mathbb{R}^2 \setminus \bar{\Omega}_0$. Using a trace inequality and Remark 4.2,

$$(4.7) \quad \begin{aligned} \|u\|_{L^2(\Gamma_R)} &\leq C \|u\|_{H^1(\Omega_c)} \quad \text{and} \\ \|\nabla u\|_{L^2(\Gamma_R)} &\leq C \|u\|_{H^2(S_\gamma)} \leq C \|u\|_{H^1(\Omega_c)}. \end{aligned}$$

It follows from (3.4) and Lemma 4.1 that

$$(4.8) \quad |\Phi(\tilde{r})| \leq Ce^{-k\text{Im}(\tilde{r})} \leq Ce^{-\alpha k|y|}.$$

This implies

$$(4.9) \quad \left(\int_{\Gamma_R} |\Phi(\tilde{r})|^2 dS_y \right)^{1/2} \leq C \left(\int_{\Gamma_R} e^{-2\alpha k R} dS_y \right)^{1/2} \leq Ce^{-\alpha_1 k R}$$

for some $0 < \alpha_1 < \alpha$.

Using Lemma 3.3, (3.4) and Lemma 4.1, we see that

$$(4.10) \quad \left| \frac{\partial \Phi(\tilde{r})}{\partial y_j} \right| = \left| \Phi'(\tilde{r}) \frac{(\tilde{x}_j - \tilde{y}_j)(-d(y_j))}{\tilde{r}} \right| \leq C |\Phi'(\tilde{r})| \leq Ce^{-\alpha k|y|}.$$

A simple computation as in (4.9) shows that

$$(4.11) \quad \|\nabla \Phi(\tilde{r})\|_{L^2(\Gamma_R)} \leq Ce^{-\alpha_1 k R}$$

for some positive α_1 . Combining (4.7), (4.9), and (4.11) gives

$$I \leq Ce^{-\alpha_1 k R} \|u\|_{H^1(\Omega_c)}.$$

Since I converges to zero as R tends to infinity, there is no contribution of the outer boundary Γ_R . Finally, we obtain (4.6) since H is the identity on Γ_0 . \square

Remark 4.4. The above theorem is a uniqueness result. It is not hard to see that the right hand side of (4.6) gives a function w which satisfies $(\tilde{\Delta} + k^2 I)w = 0$ in $\mathbb{R}^2 \setminus \bar{\Omega}_0$. The theorem shows that there is only one function satisfying $(\tilde{\Delta} + k^2 I)w = 0$ in $\mathbb{R}^2 \setminus \bar{\Omega}_0$ along with

$$w = u \text{ and } \frac{\partial w}{\partial n} = \frac{\partial u}{\partial n} \text{ on } \Gamma_0.$$

The following proposition shows the uniqueness of solutions to the Cartesian PML problem in the infinite domain (1.4).

Proposition 4.5. *The Cartesian PML problem (1.4) with $f = 0$ has only a trivial solution in $H_0^1(\Omega_c)$.*

Proof. Let \tilde{u} be a solution to (1.4) with $f = 0$ in $H_0^1(\Omega_c)$. By Theorem 4.3, \tilde{u} can be expressed in the integral formula

$$(4.12) \quad \tilde{u}(x) = \int_{\Gamma_0} \left[\tilde{u}(y) \frac{\partial \Phi(\tilde{r})}{\partial n_y} - \Phi(\tilde{r}) \frac{\partial \tilde{u}}{\partial n}(y) \right] dS_y$$

for $x \in \mathbb{R}^2 \setminus \bar{\Omega}_0$.

Define

$$(4.13) \quad u(x) = \begin{cases} \tilde{u}(x) & \text{for } x \in \bar{\Omega}_0 \setminus \bar{\Omega}, \\ \int_{\Gamma_0} \left[\tilde{u}(y) \frac{\partial \Phi(|x-y|)}{\partial n_y} - \Phi(|x-y|) \frac{\partial \tilde{u}}{\partial n}(y) \right] dS_y & \text{for } x \in \mathbb{R}^2 \setminus \bar{\Omega}_0. \end{cases}$$

Note that the transition at Ω_0 is smooth since $\Phi(|x-y|)$ coincides with $\Phi(\tilde{r})$ near Ω_0 . It follows that u satisfies (1.1) with $g = 0$. As (1.1) has unique solutions, \tilde{u} vanishes identically on $\bar{\Omega}_0 \setminus \bar{\Omega}$ and outside by (4.12). \square

We combine the sesquilinear forms in (1.4) and define

$$A_{k^2}(\cdot, \cdot) = A(\cdot, \cdot) - k^2(J \cdot, \cdot).$$

We then have the following lemma which provides stability of the PML problem on Ω_c .

Lemma 4.6. *For any real $k \neq 0$, the following two inf-sup conditions hold: For u in $H^1(\Omega_c)$,*

$$\|u\|_{H^1(\Omega_c)} \leq C \sup_{\phi \in H_0^1(\Omega_c)} \frac{|A_{k^2}(u, \phi)|}{\|\phi\|_{H^1(\Omega_c)}},$$

and

$$\|u\|_{H^1(\Omega_c)} \leq C \sup_{\phi \in H_0^1(\Omega_c)} \frac{|A_{k^2}(\phi, u)|}{\|\phi\|_{H^1(\Omega_c)}}.$$

Proof. It follows from Remark 2.6 and Theorem 4.6 of [28], that for any real $k \neq 0$, either the above two inf-sup conditions hold or there is an eigenvector corresponding to k^2 , i.e., a nonzero function $w \in H_0^1(\Omega_c)$ satisfying

$$A_{k^2}(w, \phi) = 0 \text{ for all } \phi \in H_0^1(\Omega_c).$$

The lemma follows since Proposition 4.5 prohibits such a w . \square

We now have the first main result on the solvability of the infinite domain problem (1.4).

Theorem 4.7. *Let k be real and positive and g be in $H^{1/2}(\Gamma)$. Then there exists a unique solution $\tilde{u} \in H^1(\Omega_c)$ to the problem*

$$(4.14) \quad A_{k^2}(\tilde{u}, \phi) = 0 \text{ for all } \phi \in H_0^1(\Omega_c)$$

with $\tilde{u} = g$. Moreover, $\|\tilde{u}\|_{H^1(\Omega_c)} \leq C\|g\|_{H^{1/2}(\Gamma)}$. Finally, the solution \tilde{u} decays exponentially, i.e., there exist $C > 0$ and $\alpha_1 > 0$ ($\alpha_1 \leq \alpha$ with α in Lemma 4.1) such that for $\|x\|_\infty \geq b$ and $\delta \geq b$,

$$(4.15) \quad |\tilde{u}(x)| \leq C e^{-\alpha_1 k |x|} \|g\|_{H^{1/2}(\Gamma)} \quad \text{and} \quad \|\tilde{u}\|_{H^{1/2}(\Gamma_\delta)} \leq C e^{-\alpha_1 k \delta} \|g\|_{H^{1/2}(\Gamma)}.$$

Proof. The solvability of (4.14) easily follows from Lemma 4.6 and we conclude that the problem (4.14) has a unique weak solution $\tilde{u} \in H^1(\Omega_c)$ satisfying

$$\|\tilde{u}\|_{H^1(\Omega_c)} \leq C\|g\|_{H^{1/2}(\Gamma)}.$$

By Remark 4.2, \tilde{u} is in $H^2((-3b/2, 3b/2)^2 \setminus [-b, b]^2)$ and hence it suffices to prove (4.15) for $\|x\|_\infty \geq 3b/2$ and $\delta \geq 3b/2$. This will follow from the integral formula (4.6), Lemma 4.1, and exponential decay of the fundamental solution (4.8) and (4.10). Indeed, by a Schwarz inequality and Remark 4.2 as in (4.7)

$$(4.16) \quad \begin{aligned} |\tilde{u}(x)|^2 &= \left| \int_{\Gamma_0} \tilde{u}(y) \frac{\partial \Phi(\tilde{r})}{\partial n_y} - \Phi(\tilde{r}) \frac{\partial \tilde{u}}{\partial n}(y) \, dS_y \right|^2 \\ &\leq C e^{-2\alpha k|x|} (\|\tilde{u}\|_{L^2(\Gamma_0)}^2 + \|\nabla \tilde{u}\|_{L^2(\Gamma_0)}^2) \leq C e^{-2\alpha k|x|} \|\tilde{u}\|_{H^1(\Omega_c)}^2. \end{aligned}$$

For $\gamma = b/8$ let S_γ be a γ -neighborhood of Γ_δ and set $\gamma' = b/4$. Clearly $S_\gamma \subset S_{\gamma'}$ and both are contained in the complement of $[-b, b]^2$. Applying Remark 4.2 on $S_\gamma \subset S_{\gamma'}$ and integrating (4.16) over $S_{\gamma'}$ gives

$$\begin{aligned} \|\tilde{u}\|_{H^{1/2}(\Gamma_\delta)} &\leq C\|\tilde{u}\|_{H^2(S_\gamma)} \leq C\|\tilde{u}\|_{L^2(S_{\gamma'})} \\ &\leq C\delta e^{-\alpha k\delta} \|\tilde{u}\|_{H^1(\Omega_c)} \leq C e^{-\alpha_1 k\delta} \|\tilde{u}\|_{H^1(\Omega_c)}. \end{aligned}$$

□

Remark 4.8. Theorem 4.7 holds for the adjoint problem as well since if \tilde{u} solves (4.14) then its conjugate satisfies the adjoint problem with data \bar{g} and visa versa.

5. SOLVABILITY OF THE TRUNCATED CARTESIAN PML PROBLEM

Our goal is to study the truncated Cartesian PML problem on $\Omega_\delta \setminus \bar{\Omega}$. The analysis involves an iteration associated with the solution of the exterior problem (on Ω_c) and a full truncated problem (on $\Omega_\delta = (-\delta, \delta)^2$).

We start by considering the full truncated variational problem: Find $u \in H_0^1(\Omega_\delta)$ satisfying

$$(5.1) \quad a_{k^2}(u, \theta) = \langle F, \theta \rangle \quad \text{for all } \theta \in H_0^1(\Omega_\delta).$$

Here F is a bounded linear functional on $H_0^1(\Omega_\delta)$, $\langle \cdot, \cdot \rangle$ denotes the duality pairing and

$$a_{k^2}(u, v) = \int_{\Omega_\delta} \left[\frac{d(x_2)}{d(x_1)} \frac{\partial u}{\partial x_1} \frac{\partial \bar{v}}{\partial x_1} + \frac{d(x_1)}{d(x_2)} \frac{\partial u}{\partial x_2} \frac{\partial \bar{v}}{\partial x_2} - k^2 J(x) u \bar{v} \right] dx.$$

It was shown in [28] that there is a positive constant δ_0 (cf. Remark 2.8 and Theorem 4.8 of [28]) such that the solution of (5.1) exists and is unique provided that $\delta > \delta_0$. Moreover, there is a constant C independent of δ satisfying

$$\|u\|_{H_0^1(\Omega_\delta)} \leq C\|F\|_{(H_0^1(\Omega_\delta))^*}.$$

These results hold for the adjoint problem as well. The following proposition is an immediate consequence.

Proposition 5.1. *Let g be in $H^{1/2}(\Gamma_\delta)$ with $\delta > \delta_0$. Then the problem*

$$(5.2) \quad a_{k^2}(u, \phi) = 0 \text{ for all } \phi \in H_0^1(\Omega_\delta)$$

with $u = g$ on Γ_δ has a unique solution satisfying

$$(5.3) \quad \|u\|_{H^1(\Omega_\delta)} \leq C \|g\|_{H^{1/2}(\Gamma_\delta)}.$$

The same result holds for the adjoint solution, i.e., (5.2) replaced by

$$a_{k^2}(\phi, u) = 0 \text{ for all } \phi \in H_0^1(\Omega_\delta).$$

Here C can be taken independent of δ .

The next proposition provides an inf-sup condition for the truncated PML problem (on $\Omega_\delta \setminus \bar{\Omega}$).

Proposition 5.2. *There is a constant $\tilde{\delta}_0$ and $C = C(\tilde{\delta}_0)$ such that if $\delta > \tilde{\delta}_0$,*

$$(5.4) \quad \|u\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} \leq C \sup_{\phi \in H_0^1(\Omega_\delta \setminus \bar{\Omega})} \frac{|a_{k^2}(u, \phi)|}{\|\phi\|_{H^1(\Omega_\delta \setminus \bar{\Omega})}} \text{ for all } u \in H_0^1(\Omega_\delta \setminus \bar{\Omega}).$$

In the above inequality, we have extended u and ϕ by zero to all of Ω_δ (in $a_{k^2}(u, \phi)$).

Proof. Let u be in $H_0^1(\Omega_\delta \setminus \bar{\Omega})$. To prove (5.4), we construct a solution $\phi \in H_0^1(\Omega_\delta \setminus \bar{\Omega})$ of the adjoint equation

$$a_{k^2}(\theta, \phi) = (\theta, u)_{H^1(\Omega_\delta \setminus \bar{\Omega})} \text{ for all } \theta \in H_0^1(\Omega_\delta \setminus \bar{\Omega})$$

satisfying

$$\|\phi\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} \leq C \|u\|_{H^1(\Omega_\delta \setminus \bar{\Omega})}.$$

The proposition then follows since

$$\|u\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} = \frac{|a_{k^2}(u, \phi)|}{\|u\|_{H^1(\Omega_\delta \setminus \bar{\Omega})}} \leq C \frac{|a_{k^2}(u, \phi)|}{\|\phi\|_{H^1(\Omega_\delta \setminus \bar{\Omega})}}.$$

To construct ϕ , we start by letting $\tilde{\phi} \in H_0^1(\Omega_c)$ solve the exterior problem

$$A_{k^2}(\theta, \tilde{\phi}) = (\theta, u)_{H^1(\Omega_c)} \text{ for all } \theta \in H_0^1(\Omega_c),$$

where we extend u by zero outside of $\Omega_\delta \setminus \bar{\Omega}$. By Lemma 4.6, $\tilde{\phi}$ is well defined and

$$\|\tilde{\phi}\|_{H^1(\Omega_c)} \leq C \|u\|_{H^1(\Omega_c)}.$$

Thus, we need only to construct a function χ satisfying:

$$(5.5) \quad \begin{aligned} \chi &= \tilde{\phi} \text{ on } \Gamma_\delta \text{ and } \chi = 0 \text{ on } \Gamma, \\ a_{k^2}(\theta, \chi) &= 0 \text{ for all } \theta \in H_0^1(\Omega_\delta \setminus \bar{\Omega}), \\ \|\chi\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} &\leq C \|u\|_{H^1(\Omega_\delta \setminus \bar{\Omega})}. \end{aligned}$$

Indeed, then $\phi = \tilde{\phi} - \chi$ has the desired properties.

We construct χ by iteration on Γ_δ . To start, we set $\chi_0 = \tilde{\phi}$ on Γ_δ . Clearly, $\chi_0 \in H^{1/2}(\Gamma_\delta)$. We set up a sequence $\{\chi_j\} \subset H^{1/2}(\Gamma_\delta)$ by induction. Given χ_j , we first define $w_j^1 \in H^1(\Omega_\delta)$ for $\delta > \delta_0$ in Proposition 5.1 to be the unique solution of

$$a_{k^2}(\theta, w_j^1) = 0 \text{ for all } \theta \in H_0^1(\Omega_\delta)$$

with $w_j^1 = \chi_j$ on Γ_δ . Next we define $w_j^2 \in H^1(\Omega_c)$ by

$$A_{k^2}(\theta, w_j^2) = 0 \text{ for all } \theta \in H_0^1(\Omega_c)$$

and $w_j^2 = w_j^1$ on Γ . We finally set $\chi_{j+1} = w_j^2$ on Γ_δ .

In addition, the adjoint problem of (4.14) is well-posed and its solution satisfies the exponential decay estimates of (4.15). Now, by Proposition 5.1 and Remark 4.8

$$\|w_j^1\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} \leq C \|\chi_j\|_{H^{1/2}(\Gamma_\delta)}$$

and

$$(5.6) \quad \begin{aligned} \|\chi_{j+1}\|_{H^{1/2}(\Gamma_\delta)} &= \|w_j^2\|_{H^{1/2}(\Gamma_\delta)} \leq C e^{-\alpha k \delta} \|w_j^1\|_{H^{1/2}(\Gamma)} \\ &\leq C e^{-\alpha k \delta} \|\chi_j\|_{H^{1/2}(\Gamma_\delta)}. \end{aligned}$$

We set $\tilde{\delta}_0$ by $\gamma = C e^{-\alpha k \tilde{\delta}_0} < 1$ for C in (5.6) so that

$$\|\chi_j\|_{H^{1/2}(\Gamma_\delta)} \leq \gamma^j \|\chi_0\|_{H^{1/2}(\Gamma_\delta)}.$$

Because of this, the telescoping series

$$\chi_0 = \sum_{j=0}^{\infty} (\chi_j - \chi_{j+1})$$

converges in $H^{1/2}(\Gamma_\delta)$ and the corresponding sequence

$$(5.7) \quad \sum_{j=0}^{\infty} (w_j^1 - w_j^2)$$

converges in $H^1(\Omega_\delta \setminus \bar{\Omega})$. By construction, the limit of (5.7) (which we denote by χ) equals $\tilde{\phi}$ on Γ_δ . By the definitions of w_j^1 and w_j^2 , it is also clear that each term in (5.7) vanishes on Γ and satisfies the homogeneous equation

$$a_{k^2}(\theta, w_j^1 - w_j^2) = 0 \text{ for all } \theta \in H_0^1(\Omega_\delta \setminus \bar{\Omega})$$

and so these properties hold for χ as well. Finally, by Remark 4.8 and Proposition 5.1

$$\begin{aligned} \|\chi\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} &\leq \sum_{j=0}^{\infty} \|w_j^1 - w_j^2\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} \\ &\leq C \sum_{j=0}^{\infty} \|\chi_j\|_{H^{1/2}(\Gamma_\delta)} \leq C \|\chi_0\|_{H^{1/2}(\Gamma_\delta)} \leq C \|u\|_{H^1(\Omega_\delta \setminus \bar{\Omega})}. \end{aligned}$$

Thus, χ satisfies all of the conditions of (5.5) and the proof is complete. \square

Remark 5.3. The inf-sup condition for the adjoint problem follows immediately from (5.4) and the fact that the coefficients in the forms are symmetric.

Remark 5.4. Proposition 5.2 shows that the inf-sup condition holds for any single positive real number k^2 provided that δ is large enough. In [26] we will extend this result to uniform stability for z in any compact subset K contained in the intersection of the resolvent of $-\tilde{\Delta}$ on $H^{-1}(\Omega_c)$ and the sector $S \equiv \{z \in \mathbb{C} : -2 \arg(d_0) < \arg(z) < 0\}$. This result will play a crucial role in proving convergence of resonance approximations using Cartesian PML in [26].

The following theorem shows exponential convergence of solutions of the truncated problems.

Theorem 5.5. *For $\delta > \tilde{\delta}_0$, there exists a unique solution $\tilde{u}_t \in H^1(\Omega_\delta \setminus \bar{\Omega})$ to the problem*

$$(5.8) \quad a_{k^2}(\tilde{u}_t, \phi) = 0 \text{ for all } \phi \in H_0^1(\Omega_\delta \setminus \bar{\Omega})$$

with $\tilde{u}_t = g$ on Γ and $\tilde{u}_t = 0$ on Γ_δ satisfying

$$(5.9) \quad \|\tilde{u}_t\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} \leq C \|g\|_{H^{1/2}(\Gamma)}.$$

Here C is independent of δ . In addition, if \tilde{u} is the solution to the infinite PML problem (4.14), then

$$(5.10) \quad \|\tilde{u} - \tilde{u}_t\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} \leq C e^{-\alpha_1 k \delta} \|g\|_{H^{1/2}(\Gamma)}$$

with α_1 in Theorem 4.7.

Proof. The existence and uniqueness of \tilde{u}_t and (5.9) are an immediate consequence of Proposition 5.2 and Remark 5.3.

Note that $\tilde{u} - \tilde{u}_t$ satisfies

$$\begin{aligned} a_{k^2}(\tilde{u} - \tilde{u}_t, \phi) &= 0 \text{ for all } \phi \in H_0^1(\Omega_\delta \setminus \bar{\Omega}), \\ \tilde{u} - \tilde{u}_t &= 0 \text{ on } \Gamma \text{ and } \tilde{u} - \tilde{u}_t = \tilde{u} \text{ on } \Gamma_\delta. \end{aligned}$$

Proposition 5.2 and Remark 5.3 then implies that

$$\|\tilde{u} - \tilde{u}_t\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} \leq C \|\tilde{u}\|_{H^{1/2}(\Gamma_\delta)}$$

and (5.10) follows from Theorem 4.7. □

6. FINITE ELEMENT ANALYSIS

In this section, we discuss properties of the finite element approximation of the solution \tilde{u}_t of the variational problem (5.8). As this analysis is standard, we only give a brief sketch of the arguments. For simplicity, we assume that Γ is polygonal as the errors which result from the finite element method associated with boundary approximation are well understood.

Let \mathcal{T}_h denote a partition of shape-regular triangular (or quadrilateral) meshes of $\Omega_\delta \setminus \bar{\Omega}$, and h represents the diameters of elements, e.g., $h = \max_{K \in \mathcal{T}_h} \text{diam}(K)$. Let S_h denote a subspace of $H^1(\Omega_\delta \setminus \bar{\Omega})$ consisting of piecewise polynomial finite element functions and S_h^0 denote the subset of functions in S_h which vanish on $\Gamma \cup \Gamma_\delta$. We assume that g is the trace of a function in our approximation space as the additional errors associated with boundary quadrature in the finite element method are well understood. Let \tilde{S}_h be the set of functions in S_h which coincide with g on Γ and vanish on Γ_δ . In this case, the finite element approximation to \tilde{u}_t is the function in $\tilde{u}_h \in \tilde{S}_h$ satisfying

$$a_{k^2}(\tilde{u}_h, \theta) = 0 \text{ for all } \theta \in S_h^0.$$

The unique solvability of \tilde{u}_h is a consequence of an argument of Schatz [33]. Since the real parts of the elements of H are uniformly bounded from below by a positive constant and J is bounded, the sesquilinear form $a_{k^2}(\cdot, \cdot)$ satisfies a Gårding inequality.

Given $g \in L^2(\Omega_\delta \setminus \bar{\Omega})$, let $\phi \in H_0^1(\Omega_\delta \setminus \bar{\Omega})$ be the solution to the adjoint problem:

$$a_{k^2}(\theta, \phi) = (\theta, g) \text{ for all } \theta \in H_0^1(\Omega_\delta \setminus \bar{\Omega}).$$

By the smoothness of $\tilde{\sigma}$ given by (1.3), the elliptic regularity for the adjoint problem is determined by its behavior near Γ , i.e., $\phi \in H^{1+s}(\Omega_\delta \setminus \bar{\Omega})$ for some $s > 1/2$.

Under these conditions, the technique of [33] (see, also, [34]) gives that there is a positive number h_0 such that for $h < h_0$, \tilde{u}_h is uniquely defined and satisfies

$$\|\tilde{u}_t - \tilde{u}_h\|_{H^1(\Omega_\delta \setminus \bar{\Omega})} \leq C \inf_{\phi_h \in \tilde{S}_h} \|\tilde{u}_t - \phi_h\|_{H^1(\Omega_\delta \setminus \bar{\Omega})}.$$

Remark 6.1. In contrast to earlier sections, the analysis suggested in this section leads to constants (i.e., h_0 and C above) which may depend on δ . To get control over these constants, we would need to derive a uniform bound (as a function of δ) for the elliptic regularity constant, i.e., $C(\delta)$ satisfying

$$\|\phi\|_{H^{1+s}(\Omega_\delta \setminus \bar{\Omega})} \leq C(\delta) \|g\|_{H^{-1+s}(\Omega_\delta \setminus \bar{\Omega})}.$$

This is beyond the scope of this paper.

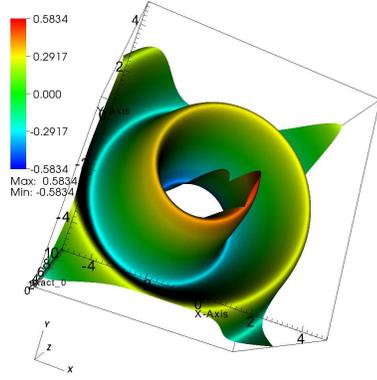
7. NUMERICAL EXPERIMENTS

As a numerical example we consider a scattering problem (1.1) with a square scatterer $\Omega = (-1, 1)^2$ in \mathbb{R}^2 with the wave number $k = 2$. The boundary condition is given by $g = e^{i\theta} H_1^1(kr)$ on Γ , where r and θ are the polar coordinates of x . Clearly, $u(x) = e^{i\theta} H_1^1(kr)$ satisfies (1.1).

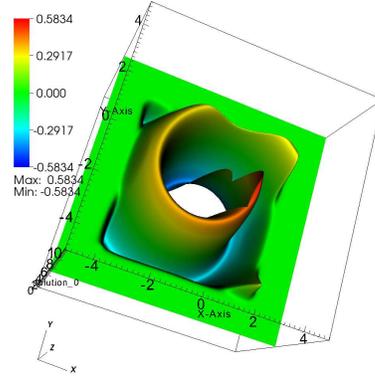
A Cartesian PML with the parameters

$$a = 3, b = 4, \sigma_0 = 1$$

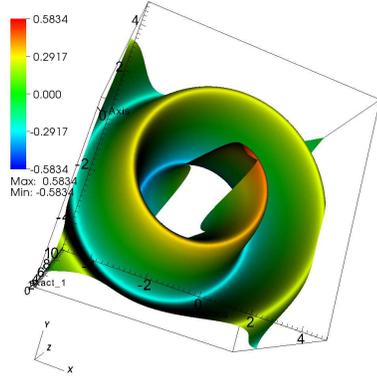
is applied to (1.1) and so we will measure the error between the finite element PML solutions and the exact one on the “region of computational interest” $[-3, 3]^2 \setminus [-1, 1]^2$.



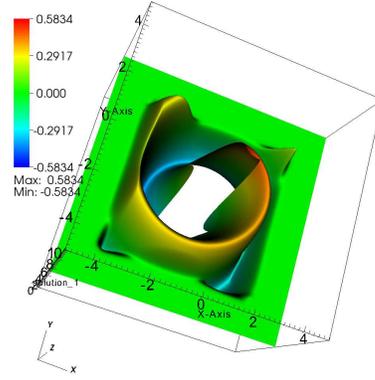
(a) Real part of the exact solution



(b) Real part of the finite element PML solution



(c) Imaginary part of the exact solution



(d) Imaginary part of the finite element PML solution

FIGURE 3. Exact solution and its finite element PML approximation

For numerical computation, the infinite domain is truncated to a finite domain $[-5, 5]^2 \setminus [-1, 1]^2$ ($\delta = 5$).

The numerical results obtained using the finite element library DEAL.II [2, 3] are given in Figure 3 and Table 1. As shown in Figure 3, the finite element PML solution is very close to the exact solution in $[-3, 3]^2 \setminus [-1, 1]^2$ and rapidly decays to zero outside. This is also illustrated in Figure 4 which shows the graphs of the real and imaginary parts of the exact solution and the finite element PML approximation at $x_2 = 2$ as functions of x_1 with $-5 \leq x_1 \leq 5$.

To further illustrate convergence of the finite element PML solutions, the errors between the interpolant of the exact solution u and the finite element PML solution \tilde{u}_h are reported in Table 1 on the region $[-3, 3]^2 \setminus [-1, 1]^2$ for different h . Note that

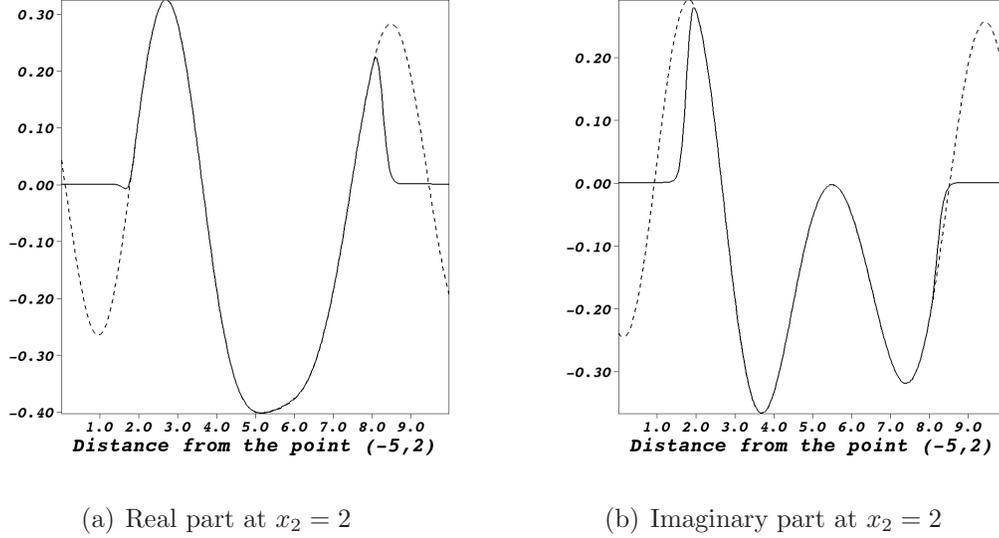


FIGURE 4. Graphs of real and imaginary parts of the exact solution (dashed curves) and the finite element PML approximation (solid curves for $h = 1/32$) at $x_2 = 2$ as functions of x_1 in $[-5, 5]$

the finite element PML solution \tilde{u}_h approximates the truncated PML solution \tilde{u}_t which cannot be analytically determined inside the layer. The table suggests first order convergence in H^1 and second order convergence in L^2 . This is not surprising because by Theorem 5.2, the truncated solution \tilde{u}_t is exponentially close to the original solution u in the region of interest $[-3, 3]^2 \setminus [-1, 1]^2$.

TABLE 1. Convergence of the real part of the finite element PML approximate solutions

| h | # dofs | real H^1 -error | | real L^2 -error | |
|------|--------|-------------------|-------|-------------------|-------|
| 1 | 240 | 1.678e+00 | ratio | 5.621e-01 | ratio |
| 1/2 | 864 | 9.791e-01 | 1.71 | 2.955e-01 | 1.90 |
| 1/4 | 3264 | 5.642e-01 | 1.74 | 1.949e-01 | 1.90 |
| 1/8 | 12672 | 2.104e-01 | 2.68 | 3.957e-02 | 4.93 |
| 1/16 | 49920 | 9.913e-02 | 2.12 | 1.042e-02 | 3.80 |
| 1/32 | 198144 | 4.866e-02 | 2.04 | 2.646e-03 | 3.94 |
| 1/64 | 789504 | 2.421e-02 | 2.01 | 6.643e-04 | 3.98 |

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9. APPENDIX A

In this section for simplicity, we use $x_j^z = x_j, y_j^z = y_j$, and $\tilde{r}^z = \tilde{r}$ without z dependency.

Proof of Lemma 3.1. We first consider the case of $\text{Im}(z) \geq 0$. Let $x \neq y$. By the mean value theorem,

$$x_j \tilde{\sigma}(x_j) - y_j \tilde{\sigma}(y_j) = \sigma(\xi_j)(x_j - y_j)$$

for some ξ_j between x_j and y_j and hence

$$(9.1) \quad \begin{aligned} \text{Re}(\tilde{x}_j - \tilde{y}_j) &= (1 + \text{Re}(z)\sigma(\xi_j))(x_j - y_j) \\ \text{Im}(\tilde{x}_j - \tilde{y}_j) &= \text{Im}(z)\sigma(\xi_j)(x_j - y_j). \end{aligned}$$

Since $\text{Re}(z) > -1/(2\sigma_M)$

$$(9.2) \quad (1 + \text{Re}(z)\sigma(\xi_j)) \geq 1/2.$$

If $x_j > y_j$ then

$$0 \leq \arg(\tilde{x}_j - \tilde{y}_j) = \tan^{-1} \frac{\text{Im}(z)\sigma(\xi_j)}{1 + \text{Re}(z)\sigma(\xi_j)} \leq \tan^{-1}(2\sigma_M \text{Im}(z)) \leq \frac{\pi}{2} - \varepsilon/2$$

for some $\varepsilon > 0$. Therefore, it follows immediately that

$$(9.3) \quad 0 \leq \arg((\tilde{x}_j - \tilde{y}_j)^2) \leq \pi - \varepsilon.$$

Clearly, (9.3) also holds when $y_j > x_j$. Now the sector $S_{0, \pi - \varepsilon} = \{\eta \in \mathbb{C} : 0 \leq \arg(\eta) \leq \pi - \varepsilon\}$ is closed under addition so it follows that

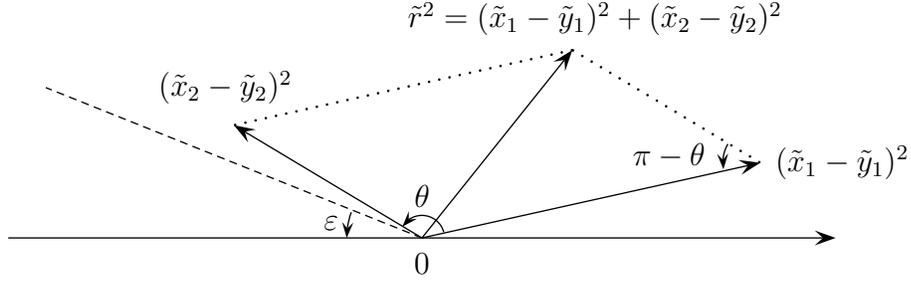
$$0 \leq \arg((\tilde{x}_1 - \tilde{y}_1)^2 + (\tilde{x}_2 - \tilde{y}_2)^2) \leq \pi - \varepsilon.$$

When $\text{Im}(z) \leq 0$, the argument is the same except both terms end up in the sector $S_{-\pi + \varepsilon, 0}$. \square

Proof of Lemma 3.3. The upper inequality is immediate from (9.1) as $|1 + z\sigma(\xi_j)|$ is uniformly bounded when z is uniformly bounded.

For the lower, we again consider the case of $\text{Im}(z) \geq 0$ as the other case is similar. We observe that

$$\begin{aligned} |\tilde{r}^2|^2 &= |(\tilde{x}_1 - \tilde{y}_1)^2 + (\tilde{x}_2 - \tilde{y}_2)^2|^2 \\ &= |\tilde{x}_1 - \tilde{y}_1|^4 + |\tilde{x}_2 - \tilde{y}_2|^4 - 2|\tilde{x}_1 - \tilde{y}_1|^2 |\tilde{x}_2 - \tilde{y}_2|^2 \cos(\pi - \theta), \end{aligned}$$

FIGURE 5. \tilde{r}^2 in the complex plane \mathbb{C}

where θ is the positive angle between $(\tilde{x}_1 - \tilde{y}_1)^2$ and $(\tilde{x}_2 - \tilde{y}_2)^2$ (See Figure 5). Since the angle θ is in $[0, \pi - \varepsilon]$ (from the previous proof), there exists a constant $C_c = C_c(\alpha)$ such that

$$(9.4) \quad -1 \leq \cos(\pi - \theta) < C_c < 1.$$

Then by a Schwarz inequality

$$\begin{aligned} |\tilde{r}^2|^2 &\geq (1 - C_c)(|\tilde{x}_1 - \tilde{y}_1|^4 + |\tilde{x}_2 - \tilde{y}_2|^4) \\ &\geq \frac{(1 - C_c)}{2^5}(|x_1 - y_1|^2 + |x_2 - y_2|^2)^2. \end{aligned}$$

For the last inequality above, we used the arithmetic-geometric mean inequality, (9.1) and (9.2). This completes the proof of the lemma. \square

10. APPENDIX B

Here we discuss modifications that are required for analysis of the Cartesian PML in \mathbb{R}^3 . The only differences between \mathbb{R}^2 and \mathbb{R}^3 are the complexified distance \tilde{r} , the fundamental solution to the Helmholtz equation and the sesquilinear form $A(\cdot, \cdot)$ associated with the variational problem of the Cartesian PML Laplace operator.

The extension of \tilde{r} in \mathbb{R}^2 to the case of \mathbb{R}^3 is straightforward. The properties of \tilde{r} in \mathbb{R}^2 in Lemma 3.1, Remark 3.2, Lemma 3.3 and Lemma 4.1 are true in the case of \mathbb{R}^3 and they are verified in the same manner as their analogues in \mathbb{R}^2 .

As for estimates for the spherical Hankel functions, since the fundamental solution to the Helmholtz equation in \mathbb{R}^3 is defined by

$$\Phi(r) = \frac{e^{ikr}}{4\pi r},$$

the inequalities analogous to (3.3) are

$$|\Phi(z)| < \frac{C}{|z|}, \quad |\Phi'(z)| < \frac{C}{|z|^2}$$

for a constant C . Here we note that $\Phi(|x|)$ and $\Phi'(|x|)$ are integrable on a neighborhood of the origin in \mathbb{R}^3 , which allows Lemma 3.6 and so Theorem 3.4 and Remark 3.8 to hold in \mathbb{R}^3 . Also, $\Phi(z)$ and $\Phi'(z)$ in \mathbb{R}^3 are given by

$$\Phi(z) = \frac{e^{ikz}}{4\pi z}, \quad \Phi'(z) = \frac{e^{ikz}}{4\pi z} \left(ik - \frac{1}{z} \right),$$

which replaces the estimates (3.4) of the fundamental solution for large $|z|$. Thus, the fundamental solution in \mathbb{R}^3 decays exponentially when the variables are stretched into the complex plane by PML.

For the application of the results in [28] and verification of interior regularity estimates (Remark 4.2) it suffices to verify the following inf-sup condition:

$$(10.1) \quad \|u\|_{H^1(D)} \leq C \sup_{v \in H^1(D)} \frac{|A(u, v) + \gamma(Ju, v)_D|}{\|v\|_{H^1(D)}}.$$

Here $(\cdot, \cdot)_D$ denotes the inner-product in $L^2(D)$. This result holds for any domain D and a fixed positive number γ . It is easy to prove this result under the (severe) restriction $0 < \arg(1 + i\sigma_M) < \pi/3$ (but with a complex constant γ). A proof of this result without this restriction was shown to us by Professor James Bramble and is given below.

Let u be in $H^1(D)$ and set $v = (1 - i\sigma_M)\bar{J}^{-1}u$. Note that because of the assumptions on $\tilde{\sigma}$, $\bar{J}^{-1}u$ is in $H^1(D)$ and its norm in $H^1(D)$ is equivalent, with constants only depending on $\tilde{\sigma}$, to that of u . Then

$$\begin{aligned} A(u, v) &= (1 + i\sigma_M) \int_D \left[d(x_1)^{-2} \left| \frac{\partial u}{\partial x_1} \right|^2 + d(x_2)^{-2} \left| \frac{\partial u}{\partial x_2} \right|^2 \right. \\ &\quad \left. + d(x_3)^{-2} \left| \frac{\partial u}{\partial x_3} \right|^2 \right] dx + b(u, u) = (1 + i\sigma_M)a(u, u) + b(u, u). \end{aligned}$$

The form $b(\cdot, \cdot)$ contains the terms where the original derivatives applied to v fall on \bar{J}^{-1} and hence

$$|b(u, u)| \leq C_1 \|\nabla u\|_{(L^2(D))^2} \|u\|_{L^2(D)}.$$

Moreover, it is easy to see that $\operatorname{Re}((1 + i\sigma_M)/d(x)^2) \geq \beta > 0$ holds for some β independent of x . Thus, for γ real and positive,

$$\begin{aligned} |A(u, v) + \gamma(Ju, v)| &\geq \operatorname{Re}[(1 + i\sigma_M)a(u, u)] + \gamma(u, u)_D - C_1 \|\nabla u\|_{(L^2(D))^2} \|u\|_{L^2(D)} \\ &\geq \beta \|\nabla u\|_{(L^2(D))^2}^2 + \gamma \|u\|_{L^2(D)}^2 - C_1 \|\nabla u\|_{(L^2(D))^2} \|u\|_{L^2(D)} \\ &\geq (\beta - C_1\eta/2) \|\nabla u\|_{(L^2(D))^2}^2 + (\gamma - C_1/(2\eta)) \|u\|_{L^2(D)}^2. \end{aligned}$$

The last inequality holds for any positive η . The inequality (10.1) with $C = 2/\beta$ follows taking $\eta = \beta/C_1$ and $\gamma = \beta/2 + C_1/(2\eta)$.

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